$\boxed{Ex}$ Consider the nonlinear system: $x' = x + y^2$
$y' = y - x$

(a) Find the equilibrium point that is not the origin.
(b) Classify the behaviour of the solution around this point.
(c) Sketch the behaviour of the solution around the equilibrium point.

a) Equilibrium points: $x + y^2 = 0$, $y - x = 0$ $\xrightarrow{x=y \text{ sub into other}}$ $y + y^2 = 0 \Rightarrow y(1+y) = 0 \Rightarrow y = 0, -1$.

Two equilibrium points $(0,0)$, and $(-1,-1)$.

Taylor Series expansions about $(x_0, y_0) = (-1,-1)$

$$F = x + y^2$$
$$\sim \underbrace{F(x_0, y_0)}_{\to 0} + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$
$$\sim 1(x - x_0) + (-2)(y - y_0)$$
$$\sim 1(x+1) - 2(y+1)$$

$F_y = 2y$ evaluate at $y_0 = -1$.

$$G = y - x$$
$$\sim \underbrace{G(x_0, y_0)}_{\to 0} + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0)$$
$$\sim -1(x+1) + 1(y+1)$$

so $$\begin{pmatrix} F \\ G \end{pmatrix} \cong \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x+1 \\ y+1 \end{pmatrix}$$ is the linearization about $(-1,-1)$.

<u>eigenvalues</u> $\det\begin{pmatrix} 1-\lambda & -2 \\ -1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow \lambda = 1 \pm \sqrt{2}$

real, one positive, one negative,
$\Rightarrow$ saddle point.

eigenvector $\lambda = 1+\sqrt{2} > 0$

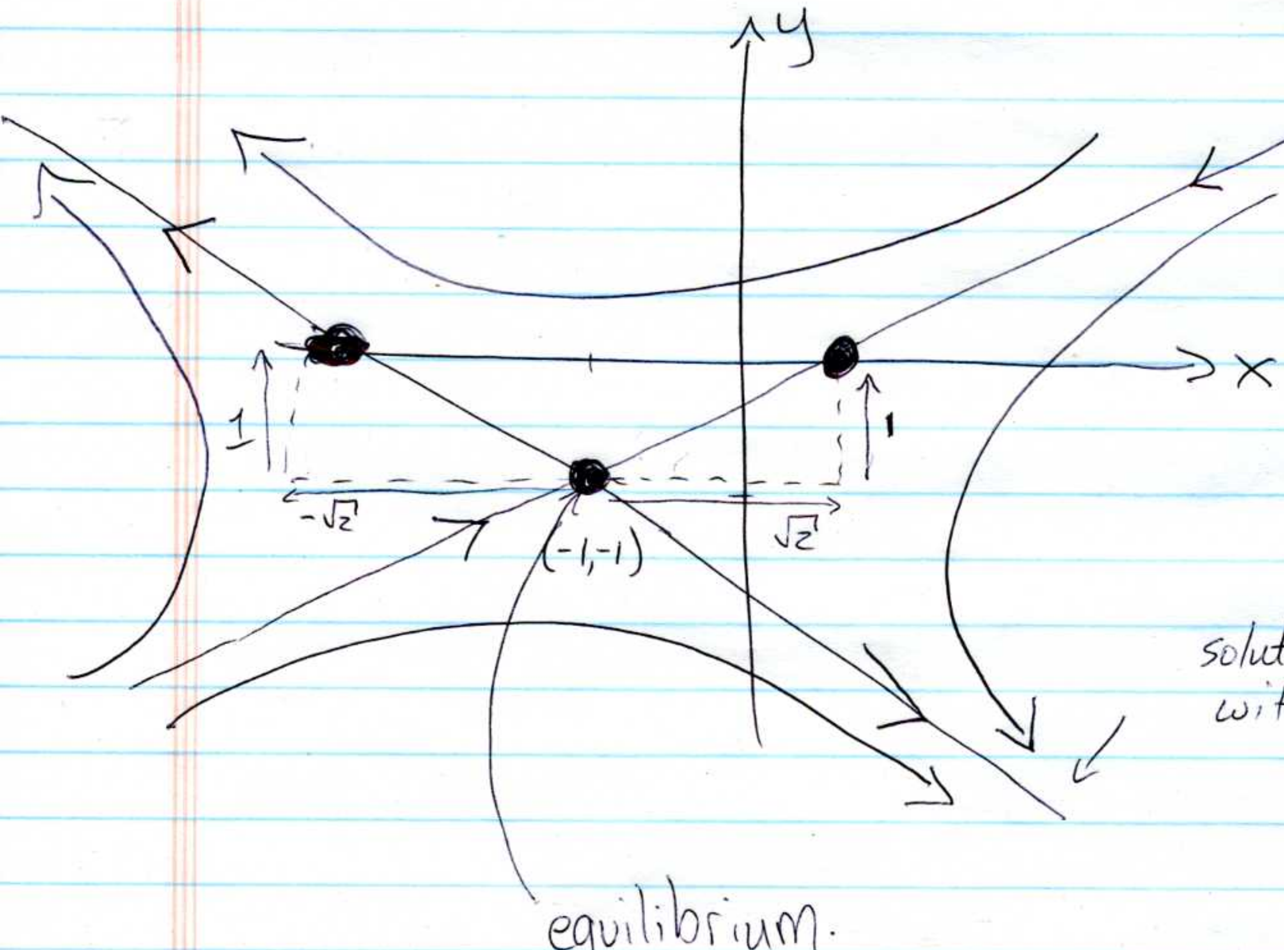
solve $\begin{pmatrix} 1-1-\sqrt{2} & -2 \\ -1 & 1-1-\sqrt{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad -\xi_1 - \sqrt{2}\,\xi_2 = 0 \quad \Rightarrow \quad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

eigenvector $\lambda = 1-\sqrt{2} < 0$

$$\begin{pmatrix} \sqrt{2} & -2 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad -\xi_1 + \sqrt{2}\,\xi_2 = 0 \quad \Rightarrow \quad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$



solution associated with $\lambda = 1-\sqrt{2}$ $\xi = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ (moves toward (1,1) since $\lambda < 0$).

solution associated with $\lambda = 1-\sqrt{2} < 0$ $\xi = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$.