

Questions

Example (5.1.1) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} (x-3)^n$.

Example (5.1.8) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$.

Example (5.1.15) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 0$ for the function $f(x) = \frac{1}{1-x}$.

Example (5.1.16) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 2$ for the function $f(x) = \frac{1}{1-x}$.

Example (5.1.21) Rewrite $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ into a sum whose generic term involves x^n .

Example (5.1.25) Rewrite $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1}$ into a sum whose generic term involves x^n .

Solutions

Example (5.1.1) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} (x-3)^n$.

Use the ratio test: if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum a_n$ converges.

Here, $a_n = (x-3)^n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} |x-3| \\ &= |x-3| \end{aligned}$$

So we require this to be less than one for convergence, which means $|x-3| < 1$. Comparing this to the form $|x-x_0| < \rho$ tells us that this series is centered at $x_0 = 3$ and has radius of convergence of $\rho = 1$.

The interval of convergence is $-1 < x-3 < 1 \rightarrow 2 < x < 4$. We would have to check the endpoints separately to find out if the series converges at $x = 2$ or $x = 4$.

Example (5.1.8) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$.

Use the root test: if $\lim_{n \rightarrow \infty} |a_n|^{1/n} < 1$ then $\sum a_n$ converges.

Here, $a_n = (n!x^n)/n^n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} |a_n|^{1/n} &= \lim_{n \rightarrow \infty} \left| \frac{n!x^n}{n^n} \right|^{1/n} \\ &= \lim_{n \rightarrow \infty} \left| \frac{n!^{1/n} x}{n} \right| \end{aligned}$$

Hmmm, don't quite know what to do next.

Try the ratio test instead: if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum a_n$ converges.

Here, $a_n = (n!x^n)/n^n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| x \left(\frac{n}{n+1} \right)^n \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n+1} \right)^n \right| \end{aligned}$$

OK, let's play a bit. Or, fire up a computer to work out the limit.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n+1} \right)^n \right| &= \lim_{m \rightarrow \infty} \left| \left(1 - \frac{1}{m} \right)^{m-1} \right| \\ &= \lim_{m \rightarrow \infty} \left| \frac{\left(1 - \frac{1}{m} \right)^m}{\left(1 - \frac{1}{m} \right)} \right| \\ &= \left| \frac{\lim_{m \rightarrow \infty} \left(1 - \frac{1}{m} \right)^m}{\lim_{m \rightarrow \infty} \left(1 - \frac{1}{m} \right)} \right| \\ &= \left| \frac{\lim_{m \rightarrow \infty} \left(1 - \frac{1}{m} \right)^m}{(1-0)} \right| \\ &= \lim_{m \rightarrow \infty} \left| \left(1 - \frac{1}{m} \right)^m \right| \\ &= \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n} \right)^n \right| = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n \end{aligned}$$

Recall from Calculus the following result:

We can express e^α as the limit of a function. Here's how:

$$\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} \\ f'(1) &= 1 \end{aligned}$$

Now, let's calculate $f'(1)$ using the definition of derivative.

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \quad (\text{relabel } h \rightarrow x) \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln(1)}{x}, \quad \ln 1 = 0 \\ &= \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \quad (\text{use logarithm law } r \ln x = \ln(x^r)) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} \\ &= \ln \lim_{x \rightarrow 0} (1+x)^{1/x} \quad (\text{interchange limit and logarithm}) \end{aligned}$$

We know this should equal 1 from our above calculation. We also know that $\ln e = 1$. Therefore, we must have:

$$e = \lim_{x \rightarrow 0} (1+x)^{1/x} \quad (\text{limit at zero})$$

Therefore,

$$\begin{aligned} e^\alpha &= \left(\lim_{x \rightarrow 0} (1+x)^{1/x} \right)^\alpha \\ &= \lim_{x \rightarrow 0} (1+x)^{\alpha/x} \quad \text{let } n = \alpha/x. \text{ As } x \rightarrow 0, n \rightarrow \infty \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n} \right)^n \\ e^{-1} &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n \end{aligned}$$

So, collecting this all together, we have:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|e^{-1}$$

So we require this to be less than one for convergence, which means $|x|e^{-1} < 1$, or $|x| < e$. Comparing this to the form $|x - x_0| < \rho$ tells us that this series is centered at $x_0 = 0$ and has radius of convergence of $\rho = e$.

The interval of convergence is $-e < x < e$. We would have to check the endpoints separately to find out if the series converges at $x = \pm e$.

Example (5.1.15) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 0$ for the function $f(x) = \frac{1}{1-x}$.

$$f(x) = \frac{1}{1-x}, \quad x_0 = 0.$$

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$(1-x)^{-1}$	$1 = 0!$
1	$1(1-x)^{-2}$	$1 = 1!$
2	$2(1-x)^{-3}$	$1 \cdot 2 = 2!$
3	$2 \cdot 3(1-x)^{-4}$	$1 \cdot 2 \cdot 3 = 3!$
4	$2 \cdot 3 \cdot 4(1-x)^{-5}$	$1 \cdot 2 \cdot 3 \cdot 4 = 4!$
5	$2 \cdot 3 \cdot 4 \cdot 5(1-x)^{-6}$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5!$
	\vdots	
n	$n!(1-x)^{-(n+1)}$	$n!$

The Taylor series is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = \sum_{n=0}^{\infty} x^n$.

Radius of convergence: Use ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \rightarrow \infty} |x| = |x| < 1$$

So radius of convergence is $\rho = 1$.

Example (5.1.16) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 2$ for the function $f(x) = \frac{1}{1-x}$.

$$f(x) = \frac{1}{1-x}, \quad x_0 = 2.$$

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$(1-x)^{-1}$	$1 = 0!$
1	$1(1-x)^{-2}$	$1(-1)^{-2} = 1!$
2	$2(1-x)^{-3}$	$1 \cdot 2(-1)^{-3} = -2!$
3	$2 \cdot 3(1-x)^{-4}$	$1 \cdot 2 \cdot 3(-1)^{-4} = 3!$
4	$2 \cdot 3 \cdot 4(1-x)^{-5}$	$1 \cdot 2 \cdot 3 \cdot 4(-1)^{-5} = -4!$
5	$2 \cdot 3 \cdot 4 \cdot 5(1-x)^{-6}$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5(-1)^{-6} = 5!$
	\vdots	
n	$n!(-1)^{n+1}(1-x)^{-(n+1)}$	$(-1)^{n+1}n!$

The Taylor series is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$.

Radius of convergence: Use ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} |(x-2)| = |x-2| < 1$$

So radius of convergence is $\rho = 1$.

Example (5.1.21) Rewrite $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ into a sum whose generic term involves x^n .

Here, we want to let $m = n - 2$. When $n = 2$, $m = 0$. Also, $n = m + 2$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

Now, relabel as $m = n$:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

Remember, n is a dummy index and that is what makes this relabelling possible.

Example (5.1.25) Rewrite $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} ka_k x^{k-1}$ into a sum whose generic term involves x^n .

Here, we want to let $n = m - 2$. When $m = 2$, $n = 0$. Also, $m = n + 2$. Also, we can bring the x inside the summation in the second summation.

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} ka_k x^{k-1} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{k=1}^{\infty} ka_k x^k$$

Now, relabel as $k = n$. Also, we need to take the $n = 0$ term out of the first sum, so both sums will begin at $n = 1$.

$$\begin{aligned} \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} ka_k x^{k-1} &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{k=1}^{\infty} ka_k x^k \\ &= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} na_n x^n \\ &= 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + na_n] x^n \end{aligned}$$

In this case, we can collect the first term in with the rest, since the quantity in the square brackets is $2a_2$ when $n = 0$. This final simplification will not always occur when we are looking for series solutions of differential equations!

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} ka_k x^{k-1} = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n] x^n$$