Questions

Example (5.2.1) Solve the differential equation $y'' - y = 0$ using a series solution about $x_0 = 0$.

Example (5.2.2) Solve the differential equation $y'' - xy' - y = 0$ using a series solution about $x_0 = 0$.

Example (5.2.21) Hermite Equation Solve the differential equation $y'' - 2xy' + \lambda y = 0$ using a series solution about $x_0 = 0$.

Example (5.3.5) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 4$ for the differential equation $y'' + 4y' + 6xy = 0$.

Example (5.3.7) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 2$ for the differential equation $(1 - x^3)y'' + 4xy' + y = 0$.

Example (5.3.11) Find the first four nonzero terms in two linearly independent series solutions about the origin to the differential equation $y'' + (\sin x)y = 0$. What do you expect the radius of convergence to be?

Solutions

Example (5.2.1) Solve the differential equation $y'' - y = 0$ using a series solution about $x_0 = 0$.

This could be solved by assuming $y = e^{rt}$, since the differential equation has constant coefficients and is linear. We will solve using series solution instead.

First, since $p(x) = 0$ and $q(x) = 1$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' - y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Relabel each term so it has an x^n:

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - a_n\right] x^n &= 0 \end{aligned}$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$(n+1)(n+2) a_{n+2} - a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

This is the recurrence relation. We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern. We will not always be able to determine a pattern!

$$\begin{aligned}
a_{n+2} &= \frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, 3, \ldots \\
a_0 &= \text{unspecified, assume not equal to zero} \\
a_1 &= \text{unspecified, assume not equal to zero} \\
a_2 &= \frac{a_0}{2} = \frac{a_0}{2!} \\
a_3 &= \frac{a_1}{6} = \frac{a_1}{3!} \\
a_4 &= \frac{a_2}{12} = \frac{a_0}{4!} \\
a_5 &= \frac{a_3}{20} = \frac{a_1}{5!}
\end{aligned}$$

The pattern in the above is $a_{2n} = a_0/(2n)!$ for even terms, and for odd terms we get $a_{2n+1} = a_1/(2n+1)!$. Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{a_1}{(2n+1)!} x^{2n+1} \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
&= a_0 \cosh x + a_1 \sinh x
\end{aligned}$$

We were able to sum the infinite series, or more precisely we recognized them as Taylor series expansions of known functions. This is what we would really like to be able to do all the time, but it is not always possible.

The a_0 and a_1 are the constants of integration which would be determined by initial conditions if we had an initial value problem. We might prefer to write the solution as $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where $y_1(x) = \cosh x$ and $y_2(x) = \sinh x$ form a fundamental set of solutions.

Example (5.2.2) Solve the differential equation $y'' - xy' - y = 0$ using a series solution about $x_0 = 0$.

This could not be solved by assuming $y = e^{rt}$, since the differential equation has variable coefficients.

First, since $p(x) = -x$ and $q(x) = -1$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
y'' - xy' - y &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n &= 0
\end{aligned}$$

Relabel each term so it has an x^n:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now we need each sum to start at the same value of n; we can achieve this by removing the $n = 0$ terms from the first and third sum:

$$\begin{aligned}
(2a_2 - a_0)x^0 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n &= 0 \\
(2a_2 - a_0)x^0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - n a_n - a_n\right] x^n &= 0
\end{aligned}$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$\begin{aligned}
2a_2 - a_0 &= 0 \\
(n+1)(n+2)a_{n+2} - n a_n - a_n &= 0, \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Notice that since the sum started at $n = 1$, the second equation is true for $n = 1, 2, 3, \ldots$.

These are the recurrence relations. Sometimes (but not always!) it is the case that the recurrence relations can be written for $n = 0, 1, 2, 3, \ldots$. We see here that $n = 0$ in the second equation gives us $2a_2 - a_0 = 0$, so the first equation is really the second with $n = 0$. A bit of algebra gives us for the recurrence relations (since $n + 1 \neq 0$):

$$(n+2)a_{n+2} - a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern.

$$\begin{aligned}
a_{n+2} &= \frac{a_n}{(n+2)}, \quad n = 0, 1, 2, 3, \ldots \\
a_0 &= c_1 \text{ unspecified, assume not equal to zero} \\
a_1 &= c_2 \text{ unspecified, assume not equal to zero} \\
a_2 &= \frac{a_0}{2} = \frac{c_1}{2} \\
a_3 &= \frac{a_1}{3} = \frac{c_2}{3} \\
a_4 &= \frac{a_2}{4} = \frac{c_1}{2 \cdot 4} \\
a_5 &= \frac{a_3}{5} = \frac{c_2}{3 \cdot 5}
\end{aligned}$$

The pattern in the above is $a_{2k} = c_1/(2{\cdot}4{\cdot}6 \cdots (2k))$ for even terms, and for odd terms we get $a_{2k+1} = c_2/(1{\cdot}3{\cdot}5{\cdot}7 \cdots (2k{+}1))$.

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\
&= c_1 \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)} + c_2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)} \\
&= c_1 y_1(x) + c_2 y_2(x) \\
y_1(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)} \\
y_2(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)}
\end{aligned}$$

The $y_1(x)$ and $y_2(x)$ are linearly independent since one is odd and the other even, so they form a fundamental set of solutions.

We would like to simplify these functions if possible, so let's see what we can do!

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdots (2n)} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (1 \cdot 2 \cdot 3 \cdots n)} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2} \right)^n \\
&= e^{x^2/2}
\end{aligned}$$

where we recognized the Taylor series $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$.

The second function is trickier. Notice that we almost have a factorial in the denominator, but we are missing $2 \cdot 4 \cdot 6 \cdots (2n) = 2(1 \cdot 2 \cdot 3 \cdots n)$:

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1} 2(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n+1)} \\
&= \sum_{n=0}^{\infty} \frac{2n! x^{2n+1}}{(2n+1)!}
\end{aligned}$$

This is an improvement, since we have removed the $\cdots$ and now have factorials. We can leave it here, but if you fire up *Mathematica* you can simplify this even further.

$$y_2(x) = \sum_{n=0}^{\infty} \frac{2n! x^{2n+1}}{(2n+1)!} = 2e^{x^2/2} \sqrt{\pi}\, \mathrm{erf}(x/2).$$

The error function $\text{erf}(x)$ is given by $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$, and it is commonly seen in probability theory.

Example (5.2.21) Hermite Equation Solve the differential equation $y'' - 2xy' + \lambda y = 0$ using a series solution about $x_0 = 0$.

This could <u>not</u> be solved by assuming $y = e^{rt}$, since the differential equation has variable coefficients.

First, since $p(x) = -2x$ and $q(x) = \lambda$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 2xy' + \lambda y &= 0 \\ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n &= 0 \end{aligned}$$

Relabel each term so it has an x^n:

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

Now we need each sum to start at the same value of n; we can achieve this by removing the $n = 0$ terms from the first and third sum:

$$\begin{aligned} (2a_2 + \lambda a_0)x^0 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=1}^{\infty} \lambda a_n x^n &= 0 \\ (2a_2 + \lambda a_0)x^0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - 2n a_n + \lambda a_n\right] x^n &= 0 \end{aligned}$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$\begin{aligned} 2a_2 + \lambda a_0 &= 0 \\ (n+1)(n+2)a_{n+2} - 2n a_n + \lambda a_n &= 0, \quad n = 1, 2, 3, \ldots \end{aligned}$$

Notice that since the sum started at $n = 1$, the second equation is true for $n = 1, 2, 3, \ldots$.

These are the recurrence relations. Sometimes (but not always!) it is the case that the recurrence relations can be written for $n = 0, 1, 2, 3, \ldots$. We see here that $n = 0$ in the second equation gives us $2a_2 - \lambda a_0 = 0$, so the first equation is really the second with $n = 0$. A bit of algebra gives us for the recurrence relations :

$$(n+1)(n+2)a_{n+2} + (\lambda - 2n)a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern.

$$\begin{aligned}
a_{n+2} &= \frac{a_n(2n-\lambda)}{(n+1)(n+2)}, \quad n = 0,1,2,3,\ldots \\
a_0 &= c_1 \text{ unspecified, assume not equal to zero} \\
a_1 &= c_2 \text{ unspecified, assume not equal to zero} \\
a_2 &= \frac{a_0(-\lambda)}{2} = \frac{c_1(-\lambda)}{2} \\
a_3 &= \frac{a_1(2-\lambda)}{2\cdot 3} = \frac{c_2(2-\lambda)}{3!} \\
a_4 &= \frac{a_2(4-\lambda)}{3\cdot 4} = \frac{c_1(-\lambda)(4-\lambda)}{4!} \\
a_5 &= \frac{a_3(6-\lambda)}{4\cdot 5} = \frac{c_2(2-\lambda)(6-\lambda)}{5!} \\
a_6 &= \frac{a_4(8-\lambda)}{5\cdot 6} = \frac{c_1(-\lambda)(4-\lambda)(8-\lambda)}{6!} \\
a_7 &= \frac{a_5(10-\lambda)}{6\cdot 7} = \frac{c_2(2-\lambda)(6-\lambda)(10-\lambda)}{7!}
\end{aligned}$$

The pattern in the above is a bit difficult to write out, but it is readily apparent there is a pattern! For even terms, the pattern is

$$\begin{aligned}
a_{2k+2} &= c_1\frac{(2\cdot 0-\lambda)(2\cdot 2-\lambda)(2\cdot 4-\lambda)(2\cdot 6-\lambda)\cdots(2(2k)-\lambda)}{(2k+2)!}, \ k = 0,1,2,3,\ldots \\
&= c_1\frac{(4\cdot 0-\lambda)(4\cdot 1-\lambda)(4\cdot 2-\lambda)(4\cdot 3-\lambda)\cdots(4k-\lambda)}{(2k+2)!}, \ k = 0,1,2,3,\ldots \\
&= c_1\frac{1}{(2k+2)!}\prod_{i=0}^{k}(4\cdot i-\lambda), k = 0,1,2,3,\ldots
\end{aligned}$$

For odd terms, the pattern is identified in terms of products using $\dfrac{(2\cdot 2-\lambda)(2\cdot 4-\lambda)(2\cdot 6-\lambda)\cdots(2\cdot 2k-\lambda)}{\prod_{i=1}^{k}(4\cdot i-\lambda)} = 1$,

$$\begin{aligned}
a_{2k+3} &= c_2\frac{(2\cdot 1-\lambda)(2\cdot 3-\lambda)(2\cdot 5-\lambda)\cdots(2(2k+1)-\lambda)}{(2k+3)!}, \ k = 0,1,2,3,\ldots \\
&= c_2\frac{(2\cdot 1-\lambda)(2\cdot 2-\lambda)(2\cdot 3-\lambda)(2\cdot 4-\lambda)(2\cdot 5-\lambda)\cdots(2\cdot 2k-\lambda)(2(2k+1)-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)}, \ k = 0,1,2,3,\ldots \\
&= c_2\frac{\prod_{i=1}^{2k+1}(2\cdot i-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)}, \ k = 0,1,2,3,\ldots
\end{aligned}$$

I have checked these patterns in the associated *Mathematica* file. That's always a good idea when you are doing some complicated simplifications!

Therefore,

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{k=0}^{\infty} a_{2k+2} x^{2k+2} + a_1 x + \sum_{k=0}^{\infty} a_{2k+3} x^{2k+3} \\
&= c_1 \left(1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - \lambda) x^{2k+2} \right) + c_2 \left(x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1} (2 \cdot i - \lambda)}{(2k+3)! \prod_{i=1}^{k} (4 \cdot i - \lambda)} x^{2k+3} \right) \\
&= c_1 y_1(x) + c_2 y_2(x) \\
y_1(x) &= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - \lambda) x^{2k+2} \\
y_2(x) &= x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1} (2 \cdot i - \lambda)}{(2k+3)! \prod_{i=1}^{k} (4 \cdot i - \lambda)} x^{2k+3}
\end{aligned}
$$

The $y_1(x)$ and $y_2(x)$ are linearly independent since one is odd and the other even, so they form a fundamental set of solutions. If we can't recognize the pattern, which we saw was a difficult process, we can instead write the first few terms in the series (usually four or five will do). If more terms are required, the coefficients can be calculated using the recursion relations.

The Hermite Polynomials

What follows is particularly of interest to physicists, since the Hermite polynomials $H_n(x)$ arise in solving the Schrödinger equation for a harmonic oscillator. However, it also shows one way in which special functions arise from differential equations, so in that sense it is of interest to all.

If λ is nonnegative even integer, then $\lambda = 2m$, and something interesting happens to our solutions. One of these solutions will become a polynomial in this case–the first if m is even, and the second if m is odd. Let's see how this happens.

Assume m is even.

$$
\begin{aligned}
y_1(x) &= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - \lambda) x^{2k+2} \\
&= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - 2m) x^{2k+2} \\
&= 1 + \sum_{k=0}^{\infty} \frac{2^{k+1}}{(2k+2)!} \prod_{i=0}^{k} (2 \cdot i - m) x^{2k+2} \\
&= 1 + \sum_{k=0}^{m/2} \frac{2^{k+1}}{(2k+2)!} \prod_{i=0}^{k} (2 \cdot i - m) x^{2k+2}
\end{aligned}
$$

where we have stopped summing at $k = m/2$ (which is an integer since m is even) since higher terms will have a factor $2 \cdot m/2 - m = 0$ in the product.

Assume m is odd.

$$\begin{aligned}
y_2(x) &= x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1}(2 \cdot i - \lambda)}{(2k+3)! \prod_{i=1}^{k}(4 \cdot i - \lambda)} x^{2k+3} \\
&= x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1}(2 \cdot i - 2m)}{(2k+3)! \prod_{i=1}^{k}(4 \cdot i - 2m)} x^{2k+3} \\
&= x + \sum_{k=0}^{(m-1)/2} \frac{2^{k+1}}{(2k+3)!} \frac{\prod_{i=1}^{2k+1}(\cdot i - m)}{\prod_{i=1}^{k}(2 \cdot i - m)} x^{2k+3}
\end{aligned}$$

The product in the numerator will have a zero factor when $2k + 1 - m = 0$. Therefore, we stopped the summing at $k = (m-1)/2$. This is an integer since m is odd.

The Hermite polynomial $H_m(x)$ is defined as the polynomial solution to the Hermite equation with $\lambda = 2m$ for which the coefficient of x^m is 2^m. The Hermite polynomials are found from flipping back and forth between y_1 and y_2, depending on which one has the terminating infinite sum, and then normalizing.

m	$H_m(x)$	$y_1(x)\|_m$	$y_2(x)\|_m$
0	1	1	–
1	$2x$	–	$x = \frac{1}{2}(2x)$
2	$-2 + 4x^2$	$1 - 2x^2 = -\frac{1}{2}(-2 + 4x^2)$	–
3	$-12x + 8x^3$	–	$x - \frac{2}{3}x^3 = -\frac{1}{12}(-12x + 8x^3)$
4	$12 - 48x^2 + 16x^4$	$1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}(12 - 48x^2 + 16x^4)$	–
5	$120x - 160x^3 + 32x^5$	–	$x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}(120x - 160x^3 + 32x^5)$

What this means is that the differential equation $y'' - 2xy' + 2ny = 0$, n an integer, has a solution $H_n(x)$, which is a polynomial, not an infinite series. The other solution is an infinite series, and can be represented by a Hypergeometric function.

In physics, this differential equation arises when solving the quantum mechanical harmonic oscillator. The solution which is an infinite series is not physical, since it leads to a quantum mechanical wavefunction which is infinite as $x \to \infty$.

Example (5.3.5) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 4$ for the differential equation $y'' + 4y' + 6xy = 0$.

The point $x_0 = 0$ is an ordinary point since

$$p(x) = 4, \qquad q(x) = 6x$$

are both analytic about $x_0 = 0$.

The radius of convergence of the series solution will be at least as large as the minimum of the radius of convergence of the series for $p(x) = 4$ and $q(x) = 6x$ about $x_0 = 0$.

Since $p(x)$ and $q(x)$ are already expanded in power series, and these series are not infinite, the radius of convergence for them is $\rho = \infty$.

Therefore, the series solution about $x_0 = 0$ must have a radius of convergence that is at least as large as $\rho = \infty$, which of course means it must be $\rho = \infty$.

A similar argument holds for $x_0 = 4$.

Example (5.3.7) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 2$ for the differential equation $(1 - x^3)y'' + 4xy' + y = 0$.

The point $x_0 = 0$ is an ordinary point since

$$p(x) = \frac{4x}{1 - x^3}, \qquad q(x) = \frac{1}{1 - x^3}$$

are both analytic about $x_0 = 0$.

Let's determine the radius of convergence of p and q without working out the Taylor series for them.

The complex poles of p and q all occur when $1 - x^3 = 0$, which means $x = 1^{1/3}$, which is the third root of unity (studied in Chapter 4). These roots are $x = -1, 1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2$.

The distance from $x_0 = 0$ to the nearest complex pole is 1 *(diagram in Mathematica file)*.

$$\text{distance} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

Therefore, the radius of convergence of the series for $p(x)$ and $q(x)$ is $\rho = 1$.

The minimum radius of convergence for the series solution about $x_0 = 0$ to the differential equation is $\rho = 1$.

The distance from $x_0 = 2$ to the nearest complex pole is $\sqrt{3}$ *(diagram in Mathematica file)*.

$$\text{distance} = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}.$$

Therefore, the radius of convergence of the series for $p(x)$ and $q(x)$ is $\rho = \sqrt{3}$.

The minimum radius of convergence for the series solution about $x_0 = 2$ to the differential equation is $\rho = \sqrt{3}$.

Example (5.3.11) Find the first four nonzero terms in two linearly independent series solutions about the origin to the differential equation $y'' + (\sin x)y = 0$. What do you expect the radius of convergence to be?

First, since $\sin x$ has a series solution about $x_0 = 0$ which converges for all x, we expect our series solution to converge for all x, which means the radius of convergence for the series solution should be $\rho = \infty$.

We need to expand the sine function, if we hope to collect powers of x.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Since $p(x) = 0$ and $q(x) = \sin x$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' + (\sin x) y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) &= 0 \end{aligned}$$

Since we want to collect powers of x to get a recurrence relation, we have two options here. We can multiply out the two infinite series, or truncate all of them. There is an example of working with the infinite series multiplied together on a different differential equation at http://cda.mrs.umn.edu/ mcquarrb/DigitalDE/Math/SSOP.nb.

Let's truncate them here, and see what happens. Assuming that we will get both solutions at once, if we want the first four nonzero terms in each we should go out to at least x^8. The manipulations that lead to the following equation are somewhat tedious, and I used *Mathematica* to perform them (they are in the associated *Mathematica* file).

Let's choose to truncate them all at $k = 12$ (I encourage to investigate this with the associated *Mathematica* file), and then we get the following:

$$
\begin{aligned}
&2a_2 + (a_0 + 6a_3)x + (a_1 + 12a_4)x^2 + \left(-\frac{a_0}{6} + a_2 + 20a_5\right)x^3 + \left(-\frac{a_1}{6} + a_3 + 30a_6\right)x^4 \\
&+ \left(\frac{a_0}{120} - \frac{a_2}{6} + a_4 + 42a_7\right)x^5 + \left(\frac{a_1}{120} - \frac{a_3}{6} + a_5 + 56a_8\right)x^6 + \left(-\frac{a_0}{5040} + \frac{a_2}{120} - \frac{a_4}{6} + a_6 + 72a_9\right)x^7 \\
&+ \left(-\frac{a_1}{5040} + \frac{a_3}{120} - \frac{a_5}{6} + a_7 + 90a_{10}\right)x^8 + \ldots = 0
\end{aligned}
$$

Make sure you keep enough terms in the expansion so that you aren't missing anything in the above. Set each coefficient of x to zero, and solve recursively for as many coefficients as you need (we were asked to get four nonzero terms in each solution).

$$
\begin{aligned}
a_0 &= a_0 \text{ unspecified, arbitrary not equal to zero} \\
a_1 &= a_1 \text{ unspecified, arbitrary not equal to zero} \\
2a_2 &= 0 \longrightarrow a_2 = 0 \\
a_0 + 6a_3 &= 0 \longrightarrow a_3 = -\frac{a_0}{6} \\
a_1 + 12a_4 &= 0 \longrightarrow a_4 = -\frac{a_1}{12} \\
-\frac{a_0}{6} + a_2 + 20a_5 &= 0 \longrightarrow a_5 = \frac{a_0}{120} \\
-\frac{a_1}{6} + a_3 + 30a_6 &= 0 \longrightarrow a_6 = \frac{a_1}{180} + \frac{a_0}{180} \\
\frac{a_0}{120} - \frac{a_2}{6} + a_4 + 42a_7 &= 0 \longrightarrow a_7 = -\frac{a_0}{5040} + \frac{a_1}{504}
\end{aligned}
$$

We can actually stop here. Notice we only used the first six terms in our expansion.

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots \\
&= a_0 + a_1 x - \frac{a_0}{6}x^3 - \frac{a_1}{12}x^4 + \frac{a_0}{120}x^5 + \frac{a_1}{180}x^6 + \frac{a_0}{180}x^6 - \frac{a_0}{5040}x^7 + \frac{a_1}{504}x^7 + \ldots \\
&= a_0\left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} - \frac{x^7}{5040} + \ldots\right) + \left(x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \ldots\right) \\
&= a_0 y_1(x) + a_1 y_2(x)
\end{aligned}
$$

The two linearly independent solutions are

$$
\begin{aligned}
y_1(x) &= 1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} - \frac{x^7}{5040} + \ldots \\
y_2(x) &= x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \ldots
\end{aligned}
$$