Questions

Example (5.5.1) Determine the solution to the differential equation $2xy'' + y' + xy = 0$ about $x_0 = 0$.

Example (5.5.3) Try to determine two solutions to the differential equation $xy'' + y = 0$ about $x_0 = 0$.

Example (5.6.1) Determine the exponents of the singularity for the differential equation $xy'' + 2xy' + 6e^x y = 0$ about $x_0 = 0$.

Example (5.6.11) Find the exponents at the singularity for all the regular singular points of the differential equation $(4 - x^2)y'' + 2xy' + 3y = 0$.

Solutions

Example (5.5.1) Determine the solution to the differential equation $2xy'' + y' + xy = 0$ about $x_0 = 0$.

Identify $p(x) = \dfrac{1}{2x}$ and $q(x) = \dfrac{1}{2}$.

Since $p(x)$ is not analytic at $x_0 = 0$, we have $x_0 = 0$ as a singular point. Since $xp(x) = \dfrac{1}{2}$ is analytic at $x_0 = 0$, we have $x_0 = 0$ as a regular singular point. Since $q(x)$ is analytic at $x_0 = 0$, we don't need to consider it.

Therefore, assume a solution looks like $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, and we will look for an indicial equation and recurrence relations.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}$$

Substitute into the differential equation

$$\begin{aligned}
2xy'' + y' + xy &= 0 \\
2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \\
\sum_{n=0}^{\infty} \left(2(n+r)(n+r-1)a_n + (n+r)a_n\right) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} &= 0 \\
r(2r-1)a_0 x^{r-1} + (1+r)(2r+1)a_1 x^r + \sum_{n=2}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} &= 0 \\
r(2r-1)a_0 x^{r-1} + (1+r)(2r+1)a_1 x^r + \sum_{n=2}^{\infty}\Big[(n+r)(2n+2r-1)a_n + a_{n-2}\Big] x^{n+r-1} &= 0
\end{aligned}$$

If this is true for all values of x, each coefficient of x must be zero, so we get the equations:

$$\begin{aligned} r(2r-1)a_0 &= 0 \\ (1+r)(2r+1)a_1 &= 0 \\ (n+r)(2n+2r-1)a_n + a_{n-2} &= 0, \quad n = 2,3,4,\ldots \end{aligned}$$

We can choose either of the first two equations from the above list as the indicial equation. Let's choose the first, so we must have $a_0 \neq 0$ and $r(2r-1) = 0$, so the roots of the indicial equation are $r = 0$ and $r = 1/2$.

For each root of the indicial equation, we can try to get a series solution, since we will get different recurrence relations.

$r = 0$:

$$\begin{aligned} a_0 &= \text{arbitrary, not equal to zero} \\ (1+0)(2(0)+1)a_1 &= 0 \longrightarrow a_1 = 0 \\ a_n &= -\frac{a_{n-2}}{n(2n-1)}, \quad n = 2,3,4,\ldots \\ a_2 &= -\frac{a_0}{2\cdot 3} \\ a_3 &= -\frac{a_1}{3\cdot 5} = 0 \\ a_4 &= -\frac{a_2}{4\cdot 7} = \frac{a_0}{2\cdot 3\cdot 4\cdot 7} \\ a_5 &= -\frac{a_3}{5\cdot 9} = 0 \\ a_6 &= -\frac{a_4}{6\cdot 11} = -\frac{a_0}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} \\ a_7 &= -\frac{a_5}{7\cdot 13} = 0 \end{aligned}$$

Therefore,

$$y(t) = a_0 x^0 \left(1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} - \frac{x^6}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} + \ldots\right).$$

Since a_0 is arbitrary, but not equal to zero, we can set $a_0 = 1$. A first solution of the differential equation is

$$y_1(t) = 1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} - \frac{x^6}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} + \ldots.$$

$r = 1/2$:

$$\begin{aligned} a_0 &= \text{arbitrary, not equal to zero} \\ (1+1/2)(2(1/2)+1)a_1 &= 0 \longrightarrow a_1 = 0 \\ a_n &= -\frac{a_{n-2}}{n(2n+1)}, \quad n = 2,3,4,\ldots \\ a_2 &= -\frac{a_0}{2\cdot 5} \\ a_3 &= -\frac{a_1}{3\cdot 7} = 0 \\ a_4 &= -\frac{a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9} \\ a_5 &= -\frac{a_3}{5\cdot 9} = 0 \\ a_6 &= -\frac{a_4}{6\cdot 13} = -\frac{a_0}{2\cdot 4\cdot 5\cdot 6\cdot 9\cdot 13} \\ a_7 &= -\frac{a_5}{7\cdot 13} = 0 \end{aligned}$$

Therefore,

$$y(t) = a_0 x^{1/2}\left(1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} - \frac{x^6}{2\cdot 4\cdot 5\cdot 6\cdot 9\cdot 13} + \dots\right).$$

Since a_0 is arbitrary, but not equal to zero, we can set $a_0 = 1$. A second solution of the differential equation is

$$y_2(t) = x^{1/2}\left(1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} - \frac{x^6}{2\cdot 4\cdot 5\cdot 6\cdot 9\cdot 13} + \dots\right).$$

If we had chosen the second equation as the indicial equation, we would get *exactly the same solutions*. Let's work it through for one solution, and see what happens.

The indicial equation $(1+r)(2r+1)a_1 = 0$ tells us that so we must have $a_1 \neq 0$ and $(1+r)(2r+1) = 0$, so the roots of the indicial equation are $r = -1$ and $r = -1/2$.

$\underline{r = -1}$:

$$\begin{aligned}
(-1)(2(-1)-1)a_0 &= 0 \longrightarrow a_0 = 0 \\
a_1 &= \text{arbitrary, not equal to zero} \\
a_n &= -\frac{a_{n-2}}{(2n-3)(n-1)}, \quad n = 2, 3, 4, \dots \\
a_2 &= -\frac{a_0}{2\cdot 3} = 0 \\
a_3 &= -\frac{a_1}{3\cdot 2} \\
a_4 &= -\frac{a_2}{3\cdot 5} = 0 \\
a_5 &= -\frac{a_3}{7\cdot 4} = \frac{a_1}{2\cdot 3\cdot 4\cdot 7} \\
a_6 &= -\frac{a_4}{9\cdot 5} = 0 \\
a_7 &= -\frac{a_5}{11\cdot 6} = -\frac{a_1}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11}
\end{aligned}$$

Therefore,

$$y(t) = a_1 x^{-1}\left(x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 7} - \frac{x^7}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} + \dots\right).$$

Since a_1 is arbitrary, but not equal to zero, we can set $a_1 = 1$. A solution of the differential equation is

$$y_1(t) = x^{-1}\left(x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 7} - \frac{x^7}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} + \dots\right) = 1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} - \frac{x^6}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 11} + \dots.$$

which is what we found before.

This is because we shifted the values of r by -1, but also shifted the a_n by $+1$, which results in exactly the same solution.

Example (5.5.3) Try to determine two solutions to the differential equation $xy'' + y = 0$ about $x_0 = 0$.

Identify $p(x) = 0$ and $q(x) = \dfrac{1}{x}$.

Since $q(x)$ is not analytic at $x_0 = 0$, we have $x_0 = 0$ as a singular point. Since $x^2 q(x) = x$ is analytic at $x_0 = 0$, we have $x_0 = 0$ as a regular singular point. Since $p(x)$ is analytic at $x_0 = 0$, we don't need to consider it.

Therefore, assume a solution looks like $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, and we will look for an indicial equation and recurrence relations.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}$$

Substitute into the differential equation

$$\begin{aligned}
xy'' + y &= 0 \\
x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} &= 0 \\
r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} &= 0 \\
r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n + a_{n-1}\Big] x^{n+r-1} &= 0
\end{aligned}$$

If this is true for all values of x, each coefficient of x must be zero, so we get the equations:

$$\begin{aligned}
r(r-1)a_0 &= 0 \\
(n+r)(n+r-1)a_n + a_{n-1} &= 0, \quad n = 1, 2, 3, 4, \ldots
\end{aligned}$$

The first equation is the indicial equation, so we must have $a_0 \neq 0$ and $r(r-1) = 0$, so the roots of the indicial equation are $r = 0$ and $r = 1$. These differ by an integer, so we might expect that we will have trouble finding two solutions. You should always choose to work with the largest root of the indicial equation first.

<u>$r = 1$:</u>

$$\begin{aligned}
a_0 &= \text{arbitrary, not equal to zero} \\
a_n &= -\frac{a_{n-1}}{n(n+1)}, \quad n = 1, 2, 3, 4, \ldots \\
a_1 &= -\frac{a_0}{1\cdot 2} \\
a_2 &= -\frac{a_1}{2\cdot 3} = \frac{a_0}{1\cdot 2\cdot 2\cdot 3} \\
a_3 &= -\frac{a_2}{3\cdot 4} = -\frac{a_0}{1\cdot 2\cdot 3\cdot 2\cdot 3\cdot 4} \\
a_n &= (-1)^n \frac{a_0}{n!(n+1)!}
\end{aligned}$$

Therefore,

$$y(t) = x^r \sum_{n=0}^{\infty} a_n x^n = a_0 x \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!(n+1)!}.$$

Since a_0 is arbitrary, but not equal to zero, we can set $a_0 = 1$. A first solution of the differential equation is

$$y_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!(n+1)!}.$$

Let's see what happens if we try to find a second solution:

<u>$r = 0$:</u>

$$\begin{aligned} a_0 &= \text{arbitrary, not equal to zero} \\ a_n &= -\frac{a_{n-1}}{n(n-1)}, \quad n = 1, 2, 3, 4, \ldots \\ a_1 &= -\frac{a_0}{1(1-1)} \end{aligned}$$

We get division by zero, so we cannot determine a second solution. We will revisit this topic again in more detail in Section 5.7, where we use reduction of order to get a second solution. This is also discussed in Section 5.6.

Example (5.6.1) Determine the exponents of the singularity for the differential equation $xy'' + 2xy' + 6e^x y = 0$ about $x_0 = 0$.

Identify $p(x) = \dfrac{2x}{x} = 2$ and $q(x) = \dfrac{6e^x}{x}$.

Since $p(x)$ is not analytic at $x_0 = 0$, we have $x_0 = 0$ as a singular point. Since $xp(x) = 2x$ and $x^2 q(x) = 6xe^x$ are both analytic at $x_0 = 0$, we have $x_0 = 0$ as a regular singular point.

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $xp(x)$ and $x^2 q(x)$:

$$\begin{aligned} xp(x) &= 2x \\ &= \sum_{n=0}^{\infty} p_n x^n \\ &= p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \end{aligned}$$

so $p_0 = 0$ (the only nonzero coefficient is $p_1 = 2$).

$$\begin{aligned} x^2 q(x) &= 6xe^x \\ &= 6x \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 6x + 6x^2 + 6\frac{x^3}{2} + \cdots \\ &= \sum_{n=0}^{\infty} q_n x^n \\ &= q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots \end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $xp(x) \sim p_0$ and $x^2 q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned} xy'' + 2xy' + 6e^x y &= 0 \\ x^2 y'' + x \cdot 2xy' + 6xe^x y &= 0 \\ x^2 y'' + x \cdot p_0 y' + q_0 y &= 0 \quad \text{associated Euler equation} \\ x^2 y'' + x \cdot (0) y' + (0) y &= 0 \\ x^2 y'' &= 0 \end{aligned} \tag{1}$$

This can be solved by assuming $y = x^r$; $y'' = r(r-1)x^{r-2}$, so substituting into Eq. (**??**),

$$\begin{aligned} x^2 r(r-1)x^r &= 0 \\ r(r-1) &= 0 \quad \text{indicial equation} \end{aligned}$$

So the exponents at the singularity are $r_1 = 0$ and $r_2 = 1$.

Example (5.6.11) Find the exponents at the singularity for all the regular singular points of the differential equation $(4-x^2)y'' + 2xy' + 3y = 0$.

First, we need to find the regular singular points.

Identify $p(x) = \dfrac{2x}{4-x^2} = \dfrac{2x}{(2-x)(2+x)}$ and $q(x) = \dfrac{3}{4-x^2} = \dfrac{3}{(2-x)(2+x)}$.

Since $p(x)$ is not analytic at $x_0 = \pm 2$, we have $x_0 = \pm 2$ as singular points. These are also the singular points for $q(x)$.

<u>Consider $x = +2$:</u>

Since $(x-2)p(x) = -\dfrac{2x}{2+x}$ and $(x-2)^2 q(x) = -\dfrac{3(x-2)}{2+x}$ are both analytic at $x_0 = +2$, we have $x_0 = +2$ as a regular singular point.

<u>Consider $x = -2$:</u>

Since $(x+2)p(x) = \dfrac{2x}{2-x}$ and $(x+2)^2 q(x) = \dfrac{3(x+2)}{2-x}$ are both analytic at $x_0 = -2$, we have $x_0 = -2$ as a regular singular point.

OK, now we need to determine the exponents at the singularity for each regular singular point.

<u>Consider $x = +2$:</u>

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $(x-2)p(x)$ and $(x-2)^2 q(x)$:

$$\begin{aligned} (x-2)p(x) &= -\frac{2x}{2+x} \\ &= -1 - \frac{1}{4}(x-2) + \frac{1}{16}(x-2)^2 + \cdots \quad \text{Taylor series about } x_0 = -2 \\ &= p_0 + p_1(x-2) + p_2(x-2)^2 + p_3(x-2)^3 + \cdots \end{aligned}$$

so $p_0 = -1$.

$$\begin{aligned} (x-2)^2 q(x) &= -\frac{3(x-2)}{2+x} \\ &= 0 - \frac{3}{4}(x-2) + \frac{3}{16}(x-2)^3 - \cdots \quad \text{Taylor series about } x_0 = -2 \\ &= q_0 + q_1(x-2) + q_2(x-2)^2 + q_3(x-2)^3 + \cdots \end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $(x-2)p(x) \sim p_0$ and $(x-2)^2q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned}
(4-x^2)y'' + 2xy' + 3y &= 0 \\
y'' + \frac{2x}{(2-x)(2+x)}y' + \frac{3}{(2-x)(2+x)}y &= 0 \\
(x-2)^2y'' + (x-2)\cdot(x-2)\frac{2x}{(2-x)(2+x)}y' + (x-2)^2\frac{3}{(2-x)(2+x)}y &= 0 \\
(x-2)^2y'' - (x-2)\cdot\frac{2x}{2+x}y' - \frac{3(x-2)}{2+x}y &= 0 \\
(x-2)^2y'' + (x-2)p_0y' + q_0y &= 0 \quad \text{associated Euler equation} \\
(x-2)^2y'' + (x-2)(-1)y' + (0)y &= 0 \\
(x-2)^2y'' - (x-2)y' &= 0 \qquad (2)
\end{aligned}$$

This can be solved by assuming $y = (x-2)^r$; $y' = r(x-2)^{r-1}$, $y'' = r(r-1)(x-2)^{r-2}$, so substituting into Eq. (**??**),

$$\begin{aligned}
(x-2)^2r(r-1)(x-2)^{r-2} - (x-2)r(x-2)^{r-1} &= 0 \\
r(r-1) - r &= 0 \quad \text{indicial equation} \\
r(r-2) &= 0
\end{aligned}$$

So the exponents at the singularity $x_0 = -2$ are $r_1 = 0$ and $r_2 = 2$.

<u>Consider $x = -2$:</u>

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $(x+2)p(x)$ and $(x+2)^2q(x)$:

$$\begin{aligned}
(x+2)p(x) &= \frac{2x}{2-x} \\
&= -1 + \frac{1}{4}(x+2) + \frac{1}{16}(x+2)^2 + \cdots \quad \text{Taylor series about } x_0 = +2 \\
&= p_0 + p_1(x+2) + p_2(x+2)^2 + p_3(x+2)^3 + \cdots
\end{aligned}$$

so $p_0 = -1$.

$$\begin{aligned}
(x+2)^2q(x) &= \frac{3(x+2)}{2-x} \\
&= 0 + \frac{3}{4}(x+2) + \frac{3}{16}(x+2)^3 - \cdots \quad \text{Taylor series about } x_0 = +2 \\
&= q_0 + q_1(x+2) + q_2(x+2)^2 + q_3(x+2)^3 + \cdots
\end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $(x+2)p(x) \sim p_0$ and $(x+2)^2q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned}
(4-x^2)y'' + 2xy' + 3y &= 0 \\
y'' + \frac{2x}{(2-x)(2+x)}y' + \frac{3}{(2-x)(2+x)}y &= 0 \\
(x+2)^2y'' + (x+2)\cdot(x+2)\frac{2x}{(2-x)(2+x)}y' + (x+2)^2\frac{3}{(2-x)(2+x)}y &= 0 \\
(x+2)^2y'' + (x+2)\cdot\frac{2x}{2-x}y' + \frac{3(x+2)}{2-x}y &= 0 \\
(x+2)^2y'' + (x+2)p_0y' + q_0y &= 0 \quad \text{associated Euler equation} \\
(x+2)^2y'' + (x+2)(-1)y' + (0)y &= 0 \\
(x+2)^2y'' - (x+2)y' &= 0 \qquad (3)
\end{aligned}$$

This can be solved by assuming $y = (x+2)^r$; $y' = r(x+2)^{r-1}$, $y'' = r(r-1)(x+2)^{r-2}$, so substituting into Eq. (**??**),

$$\begin{aligned} (x+2)^2 r(r-1)(x+2)^{r-2} - (x+2)r(x+2)^{r-1} &= 0 \\ r(r-1) - r &= 0 \quad \text{indicial equation} \\ r(r-2) &= 0 \end{aligned}$$

So the exponents at the singularity $x_0 = -2$ are $r_1 = 0$ and $r_2 = 2$.

If we can remember the following form, we can get the indicial equation directly from $F(r) = r(r-1) + p_0 r + q_0$, which is the from of the indicial equation for the associated Euler equation. If we forget it, we can use the process described in the solutions to create and solve the associated Euler equation.