Orthoganality — It's all about using trig identities! Details change for different examples.

CASE: Heat Conduction in bars: one radiating end.

Condition: $\tan(\mu_n L) = -\mu_n/K$ determines value of separation constant μ.

Evaluate: $\displaystyle\int_0^L \sin \mu_m x \sin \mu_n x \, dx$

Use $\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$

$$\int_0^L \sin \mu_m x \sin \mu_n x \, dx = \frac{1}{2}\int_0^L \left[\cos(\mu_m-\mu_n)x - \cos(\mu_m+\mu_n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(\mu_m-\mu_n)x}{\mu_m-\mu_n} - \frac{\sin(\mu_m+\mu_n)x}{\mu_m+\mu_n}\right]_0^L$$

$$= \frac{1}{2}\left[\frac{\sin(\mu_m-\mu_n)L}{\mu_m-\mu_n} - \frac{\sin(\mu_m+\mu_n)L}{\mu_m+\mu_n}\right]$$

$$= \frac{1}{2(\mu_m^2-\mu_n^2)}\left[(\mu_m+\mu_n)\sin(\mu_m-\mu_n)L - (\mu_m-\mu_n)\sin(\mu_m+\mu_n)L\right]$$

use $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

$$= \frac{1}{2(\mu_m^2-\mu_n^2)}\Big[\cancel{\mu_m \sin \mu_m L \cos \mu_n L} + \mu_n \sin \mu_m L \cos \mu_n L$$
$$- \mu_m \cos \mu_m L \sin \mu_n L - \cancel{\mu_n \cos \mu_m L \sin \mu_n L}$$
$$- \cancel{\mu_m \sin \mu_m L \cos \mu_n L} + \mu_n \sin \mu_m L \cos \mu_n L$$
$$- \mu_m \cos \mu_m L \sin \mu_n L + \cancel{\mu_n \cos \mu_m L \sin \mu_n L}\Big]$$

$$\int_0^L \sin \mu_m x \sin \mu_n x \, dx$$

$$= \frac{1}{\mu_m^2 - \mu_n^2} \left[\mu_n \sin \mu_m L \cos \mu_n L - \mu_m \cos \mu_m L \sin \mu_n L \right]$$

Now, use $\tan(\mu_n L) = \dfrac{-\mu_n}{K}$.

$$\Rightarrow \mu_n = -K \frac{\sin(\mu_n L)}{\cos(\mu_n L)}$$

$$= \frac{1}{\mu_m^2 - \mu_n^2} \left[-K \sin(\mu_m L) \sin(\mu_n L) + K \sin(\mu_m L) \sin(\mu_n L) \right]$$

$$= 0$$

Note $\int_0^L \sin \mu_m x \sin \mu_n x \, dx = 0$ if $m \neq n$.

If $m = n$, we would have the indeterminaint form $\frac{0}{0}$.

In this case, $m = n$

$$\int_0^L \sin^2(\mu_n x) \, dx = \left[\frac{x}{2} - \frac{\sin(2\mu_n x)}{4\mu_n} \right]_0^L$$

$$= \frac{L}{2} - \frac{\sin(2\mu_n L)}{4\mu_n}$$

How is this used?

When working through the bar with radiating end case, you arrive at

We need to figure out c_n!

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-c^2 \mu_n^2 t} \sin \mu_n x$$

At $t=0$, $u(x,0) = f(x)$, so we get:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \mu_n x .$$

multiply by $\sin \mu_m x$ and integrate over length of the bar.

$$\int_0^L f(x) \sin \mu_m x \, dx = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^L \sin \mu_n x \sin \mu_m x \, dx}$$

This is zero unless $m=n$. So it selects $n=m$ from the infinite sum.

$$\int_0^L f(x) \sin \mu_m x \, dx = c_m \int_0^L \sin^2(\mu_m x) \, dx$$

$$\Rightarrow \quad c_m = \frac{1}{\int_0^L \sin^2(\mu_m x) \, dx} \int_0^L f(x) \sin \mu_m x \, dx$$

$$= \frac{1}{\frac{L}{2} - \frac{\sin(2\mu_m L)}{4\mu_m}} \int_0^L f(x) \sin \mu_m x \, dx$$

$$c_m = \frac{4\mu_m}{2\mu_m L - \sin(2\mu_m L)} \int_0^L f(x) \sin(\mu_m x) \, dx$$