Example Solve the partial differential equation

$$-y\, u_x(x,y) + x\, u_y(x,y) = 0, \tag{1}$$

subject to the condition

$$u(0,y) = \cos y^2.$$

Solution The partial differential equation given can be rewritten as follows:

$$\nabla u(x,y) \cdot \langle -y, x\rangle = 0, \tag{2}$$

where $\nabla = \langle \partial/\partial x, \partial/\partial y\rangle$ and $\langle -y, x\rangle$ is a vector in the direction $-y\mathbf{i} + x\mathbf{j}$ at the point (x,y). I have chosen my coordinate system to be the right handed cartesian xyz-coordinates, where vector $\mathbf{i}$ is a unit vector in the x direction, $\mathbf{j}$ is a unit vector in the y direction, $\mathbf{k}$ is a unit vector in the z direction.

Geometrically, the solution to Eq. (2) $z = u(x,y)$ will be a surface in the xyz-coordinate system. Equation (2) is a directional derivative, $D_{\mathbf{v}}u(x,y) = \nabla u(x,y) \cdot \mathbf{v}$, and so Equation (2) tells us that the rate of change of the function $u(x,y)$ in the direction $\langle -y, x\rangle$ at the point (x,y) is zero.

This means the function must be constant, $u(x,y) = c_1$, in this direction. This is just a *level curve* of the function $u(x,y)$. All solutions must have this form to be constant in this direction. Functions without this form will not satisfy the partial differential equation.

The vector $\langle -y, x\rangle$ is a *direction field* (see Figure 1). It is the direction field associated with the ordinary differential equation

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{3}$$

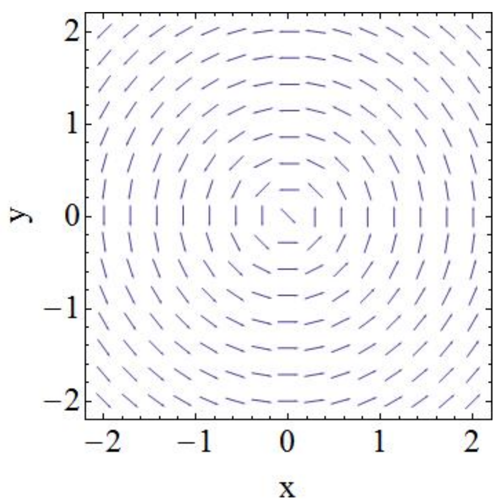


Figure 1: Along any solution curve to this direction field, the solution to the pde will be constant, $u(x,y) = c_1$.

We must solve the ordinary differential equation given in Eq. (3), since we know our solution to the partial differential equation $u(x, y)$ must be constant along curves which satisfy the ode.

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{x}{y} \\
y\,dy &= -x\,dx \\
\int y\,dy &= -\int x\,dx \\
\frac{y^2}{2} &= -\frac{x^2}{2} + c_3 \\
c_2 &= y^2 + x^2
\end{aligned} \tag{4}$$

Equation (4) give the solution curves that are represented in the direction field. They are shown against the direction field in Fig. 2.

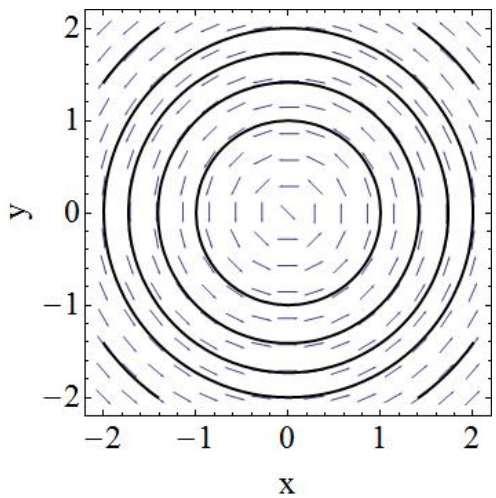


Figure 2: The solution curves $c_2 = y^2 + x^2$ against the direction field.

Getting to this point is the hard part. The constant c_1 can be expressed in terms of the constant c_2 using some function f:

$$\begin{aligned}
z = u(x, y) &= c_1 \\
&= f(c_2) \\
&= f(y^2 + x^2)
\end{aligned} \tag{5}$$

This embeds the fact that the solution $u(x, y)$ is constant in the direction $\langle 1, -x/y \rangle$ into our solution. You should check that regardless of what f is, the pde in Eq. (6) is satisfied.

We have shown that $u(x, y) = f(y^2 + x^2)$. At this point, we don't know what the function f is–it could be almost anything. In Fig. 3, I've plotted some different solutions $u(x, t)$ for different functions f.

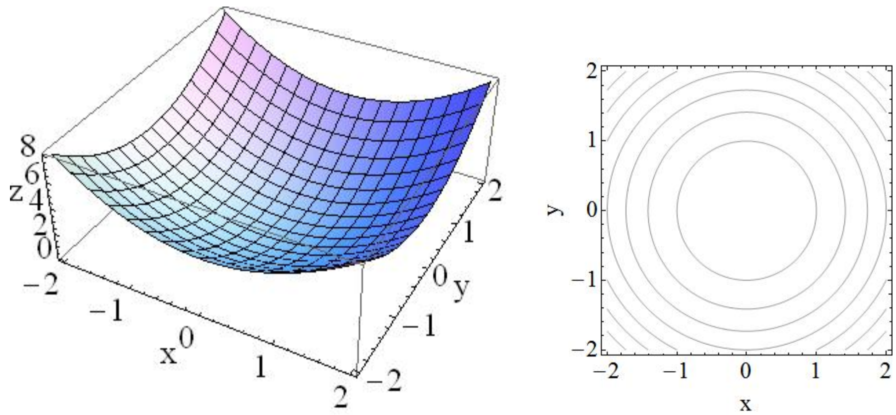
Notice in Fig. 3 that the contour plots of the surface all look the same, and all look like the direction field we found in Fig 1. This is not a coincidence, this is the geometric concept behind the method of characteristics.

What remains is to determine which function f satisfies the initial condition we were given,

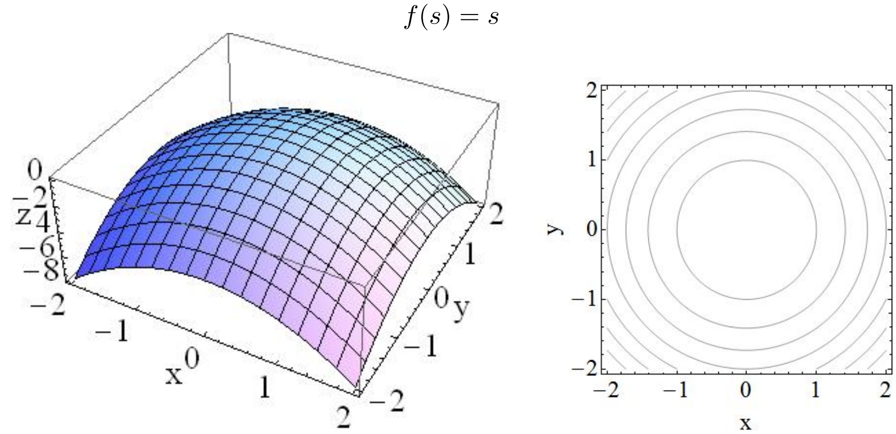
$$u(0, y) = \cos y^2.$$

We can do this by inspection,

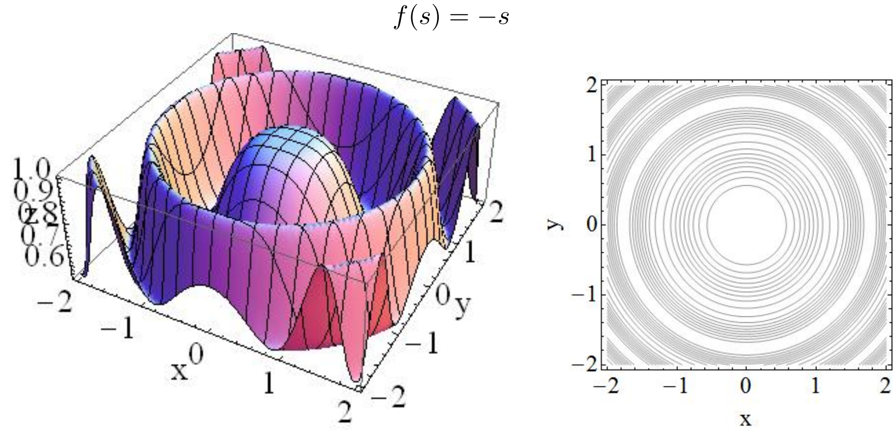
$$u(0, y) = f((y^2 + 0^2) = f(y^2) = \cos y^2$$



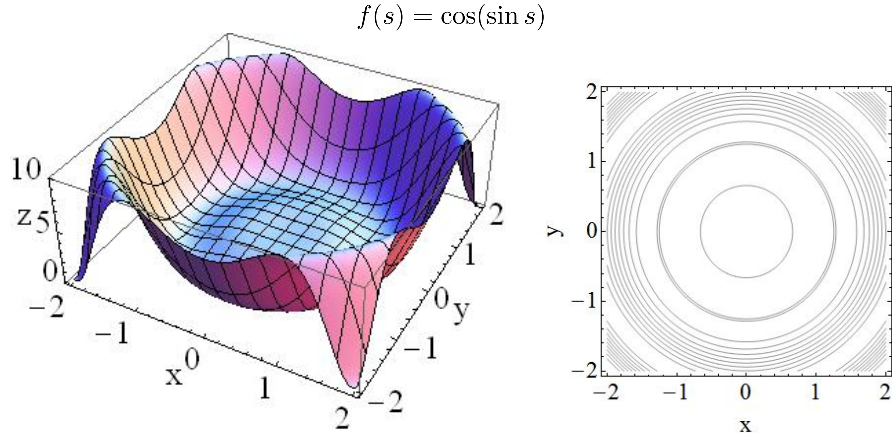
$f(s) = s$



$f(s) = -s$



$f(s) = \cos(\sin s)$



$f(s) = e^{-s^2} \cos s - s \sin s + s$

Figure 3: The function $u(x, y) = f(y^2 + x^2)$ which satisfies the given pde for different functions f.

So we see that we want $f(x) = \cos x$ to satisfy the condition given. Therefore, the solution to the given pde subject to the given condition is

$$u(x,t) = \cos(y^2 + x^2).$$

Geometrically, this means we are picking the surface that when intersected with the plane $x = 0$ gives the space curve $z = \cos y^2$. This is illustrated in Fig. 4.

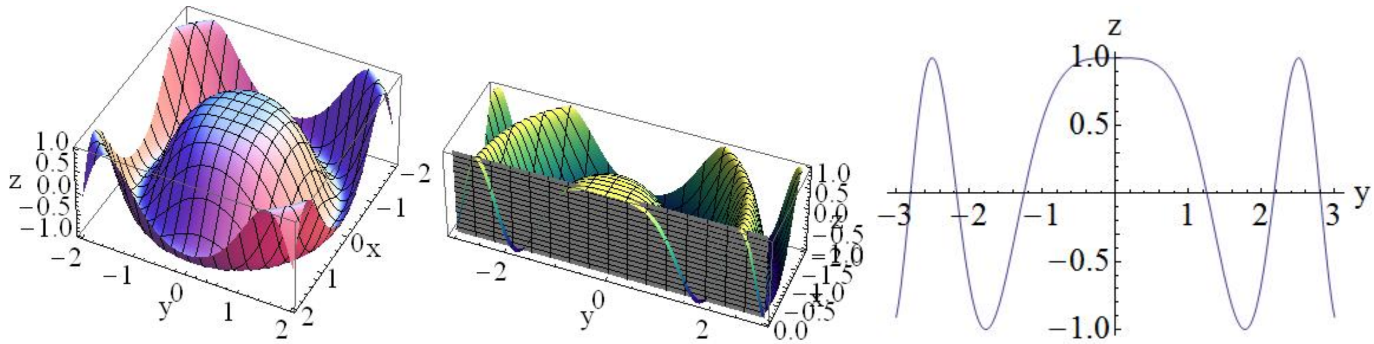


Figure 4: The function $u(x,y) = \cos(y^2 + x^2)$ which is the solution to the given boundary value problem, and the solution intersected with the plane $x = 0$ to show that it satisfies the initial condition $z = \cos y^2$ (shown on the right).

The previous example was nice in that we could explicitly solve for the solution to the direction field. We can still use the method of characteristics when the direction field involves our unknown solution $u(x,y)$, as the following example shows.

Example Solve the partial differential equation

$$u_x + A(u)\, u_y = 0, \tag{6}$$

subject to the condition

$$u(x,0) = g(x).$$

Solution The partial differential equation given can be rewritten as follows:

$$\nabla u(x,y) \cdot \langle 1, A(u) \rangle = 0, \tag{7}$$

where $\nabla = \langle \partial/\partial x, \partial/\partial y \rangle$ and $\langle 1, A(u) \rangle$ is a vector in the direction $\mathbf{i} + A(u)\mathbf{j}$ at the point (x,y). I have chosen my coordinate system to be the right handed cartesian xyz-coordinates, where vector $\mathbf{i}$ is a unit vector in the x direction, $\mathbf{j}$ is a unit vector in the y direction, $\mathbf{k}$ is a unit vector in the z direction.

Geometrically, the solution to Eq. (7) $z = u(x,y)$ will be a surface in the xyz-coordinate system. Equation (7) is a directional derivative, $D_{\mathbf{v}} u(x,y) = \nabla u(x,y) \cdot \mathbf{v}$, and so Equation (7) tells us that the rate of change of the function $u(x,y)$ in the direction $\langle 1, A(u) \rangle$ at the point (x,y) is zero.

This means the function must be constant, $u(x,y) = c_1$, in this direction. This is just a *level curve* of the function $u(x,y)$. All solutions must have this form to be constant in this direction. Functions without this form will not satisfy the partial differential equation.

The vector $\langle 1, A(u) \rangle$ is a *direction field*. It is the direction field associated with the ordinary differential equation

$$\frac{dy}{dx} = A(u). \tag{8}$$

Before we solve Eq. (8), we need to make some simplifications. Since the solution to the pde we seek is a constant on curves which satisfy the ode, we have

$$\begin{aligned} u(x,y) &= c_1 \\ &= u(x_0, 0) \\ &= g(x_0) \end{aligned} \tag{9}$$

where the point $(x_0, 0)$ is a constant. We chose to work with the point $(x_0, 0)$ since that allowed us to relate the solution $u(x, y)$ to the initial condition, g.

We now must solve the ordinary differential equation given in Eq. (8). Equation (9) allows us to integrate the differential equation without knowing the form of $u(x, y)$. We will do the integration as a definite integral between the points $(x_0, 0)$ and (x, y). This is like integrating from the initial point to the final point.

$$\begin{aligned} \frac{dy}{dx} &= A\big(u(x,y)\big) \\ &= A\big(g(x_0)\big) \\ y\,dy &= A\big(g(x_0)\big)\,dx \\ \int_0^y y\,dy &= A\big(g(x_0)\big)\int_{x_0}^x dx \\ y &= A\big(g(x_0)\big)(x - x_0) \end{aligned} \tag{10}$$

These are the characteristic curves of the pde. They are straight lines with slope given by $A(g(x_0))$.

This is as far as we can go without knowing the specific form of g or A, so we will start to look at some specific examples.

The solution to the boundary value problem is given by $u(x, y) = g(x_0)$ where x_0 is found by solving Eq. (10).

Example Solve the partial differential equation

$$u_x + [\ln u]^{-1}\, u_y = 0,$$

subject to the condition

$$u(x, 0) = e^x.$$

Solution First, we should identify the functions as they relate to our previous analysis.

$$g(x) = e^x, \quad A(u) = [\ln u]^{-1}.$$

Solving Eq. (10) for x_0 we find

$$\begin{aligned} y &= A\big(g(x_0)\big)(x - x_0) \\ &= \frac{1}{\ln e^{x_0}}(x - x_0) \\ &= \frac{1}{x_0}(x - x_0) \\ x_0 y &= x - x_0 \\ x_0 &= \frac{x}{y+1} \end{aligned}$$

The solution to the boundary value problem we were given is

$$\begin{aligned} u(x,y) &= g(x_0) \\ &= e^{x_0} \\ &= e^{x/(y+1)} \end{aligned}$$

Example Solve the partial differential equation

$$u_x + \frac{1}{1+u}\, u_y = 0, \tag{11}$$

subject to the condition

$$u(x, 0) = x^2.$$

Solution In this example we will work through the method from first principles.

The partial differential equation given can be rewritten as follows:

$$\nabla u(x,y)\cdot\langle 1,(1+u)^{-1}\rangle = 0, \tag{12}$$

where $\nabla = \langle \partial/\partial x, \partial/\partial y\rangle$ and $\langle 1,(1+u)^{-1}\rangle$ is a vector in the direction $\mathbf{i}+(1+u)^{-1}\mathbf{j}$ at the point (x,y). I have chosen my coordinate system to be the right handed cartesian xyz-coordinates, where vector $\mathbf{i}$ is a unit vector in the x direction, $\mathbf{j}$ is a unit vector in the y direction, $\mathbf{k}$ is a unit vector in the z direction.

Geometrically, the solution to Eq. (12), $z = u(x,y)$, will be a surface in the xyz-coordinate system. Equation (12) is a directional derivative, $D_{\mathbf{v}}u(x,y) = \nabla u(x,y)\cdot\mathbf{v}$, and so Equation (12) tells us that the rate of change of the function $u(x,y)$ in the direction $\langle 1,(1+u)^{-1}\rangle$ at the point (x,y) is zero.

This means the function must be constant, $u(x,y) = c_1$, in this direction. This is just a *level curve* of the function $u(x,y)$. All solutions must have this form to be constant in this direction. Functions without this form will not satisfy the partial differential equation.

The vector $\langle 1,(1+u)^{-1}\rangle$ is a *direction field*. It is the direction field associated with the ordinary differential equation

$$\frac{dy}{dx} = \frac{1}{1+u}. \tag{13}$$

Before we solve Eq. (13), we need to make some simplifications. Since the solution to the pde we seek is a constant on curves which satisfy the ode, we have

$$\begin{aligned} u(x,y) &= c_1 \\ &= u(x_0,0) \\ &= x_0^2 \end{aligned} \tag{14}$$

where the point $(x_0,0)$ is a constant. We chose to work with the point $(x_0,0)$ since that allowed us to relate the solution $u(x,y)$ to the initial condition, $u(x,0)$.

We now must solve the ordinary differential equation given in Eq. (13). Equation (14) allows us to integrate the differential equation without knowing the form of $u(x,y)$. We will do the integration as a definite integral between the points $(x_0,0)$ and (x,y).

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{1+u(x,y)} \\ &= \frac{1}{1+x_0^2} \\ y\,dy &= \frac{1}{1+x_0^2}\,dx \\ \int_0^y y\,dy &= \frac{1}{1+x_0^2}\int_{x_0}^x dx \\ y &= \frac{1}{1+x_0^2}(x-x_0) \end{aligned} \tag{15}$$

Now we solve Eq. (15) for x_0. It is a simple matter to rearrange the equation so you can use the quadratic formula, and we find:

$$x_0 = \frac{-1 \pm \sqrt{1+4xy-4y^2}}{2y}.$$

Which x_0 should we choose? Let's get the two solutions and then pick the correct one. Using $u(x,y) = x_0^2$, we find

$$\begin{aligned} u_1(x,y) &= \left(\frac{-1+\sqrt{1+4xy-4y^2}}{2y}\right)^2 \\ u_2(x,y) &= \left(\frac{-1-\sqrt{1+4xy-4y^2}}{2y}\right)^2 \end{aligned}$$

Only one of these will satisfy $\lim_{y\to 0} u(x,y) = x^2$. Notice that

$$\lim_{y\to 0} u_2(x,y) = \lim_{y\to 0} \frac{\left(-1 - \sqrt{1+4xy-4y^2}\right)^2}{4y^2} \to \infty$$

So we choose as our solution

$$u(x,y) = \frac{\left(-1 + \sqrt{1+4xy-4y^2}\right)^2}{4y^2}$$

You can check that this function has $\lim_{y\to 0} u(x,y) = x^2$.