Initial Conditions

Applying the initial condition yields the following:

$$\begin{aligned} f(r,\theta) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})A_{mn}\left(\cos(m\theta)\bar{A}_{mn} + \sin(m\theta)\bar{B}_{mn}\right) \\ &= \sum_{n=1}^{\infty} J_0(r\lambda_{0n})A_{0n}\bar{A}_{0n} \\ &+ \sum_{m=1}^{\infty}\left[\left(\sum_{n=1}^{\infty} J_m(r\lambda_{mn})A_{mn}\bar{A}_{mn}\right)\cos(m\theta) + \left(\sum_{n=1}^{\infty} J_m(r\lambda_{mn})A_{mn}\bar{B}_{mn}\right)\sin(m\theta)\right] \end{aligned} \tag{1}$$

So,if r is held fixed, this is a Fourier series expansion of $f(r,\theta)$. That means we know, using the expressions for the Fourier coefficients of a function with period 2π, that

$$\begin{aligned} a_0(r) &= \frac{1}{2\pi}\int_0^a f(r,\theta)\,d\theta = \sum_{n=1}^{\infty} J_0(r\lambda_{0n})A_{0n}\bar{A}_{0n} \\ a_m(r) &= \frac{1}{\pi}\int_0^a f(r,\theta)\cos(m\theta)\,d\theta = \sum_{n=1}^{\infty} J_m(r\lambda_{mn})A_{mn}\bar{A}_{mn} \\ b_m(r) &= \frac{1}{\pi}\int_0^a f(r,\theta)\sin(m\theta)\,d\theta = \sum_{n=1}^{\infty} J_m(r\lambda_{mn})A_{mn}\bar{B}_{mn} \end{aligned}$$

The following calculation, which employs the orthogonality relations for Bessel functions, shows how we can calculate $A_{0k}\bar{A}_{0k}$:

$$\begin{aligned} \Big[a_0(r) &= \sum_{n=1} J_0(r\lambda_{0n})A_{0n}\bar{A}_{0n}\Big] J_0(r\lambda_{0k})r \\ \int_0^a a_0(r)J_0(r\lambda_{0k})r\,dr &= \sum_{n=1} A_{0n}\bar{A}_{0n}\int_0^a J_0(r\lambda_{0n})J_0(r\lambda_{0k})r\,dr \\ &= \sum_{n=1} A_{0n}\bar{A}_{0n}\left(\frac{a^2}{2}J_1^2(\alpha_{0k})\delta_{kn}\right) \\ &= A_{0k}\bar{A}_{0k}\left(\frac{a^2}{2}J_1^2(\alpha_{0k})\right) \\ A_{0k}\bar{A}_{0k} &= \frac{2}{a^2J_1^2(\alpha_{ok})}\int_0^a a_0(r)J_0(r\lambda_{0k})r\,dr \end{aligned}$$

where $a_0(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(r,\theta)\,d\theta$. If we like, we can combine these to finally get

$$A_{0k}\bar{A}_{0k} = \frac{1}{\pi a^2 J_1^2(\alpha_{0k})}\int_0^a\int_0^{2\pi} f(r,\theta)J_0(r\lambda_{0k})r\,d\theta dr$$

Similar calculations yield the following results:

$$\begin{aligned}
A_{mk}\bar{A}_{mk} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mk})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos(m\theta)J_m(r\lambda_{mk})r\,d\theta dr \\
A_{mk}\bar{B}_{mk} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mk})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin(m\theta)J_m(r\lambda_{mk})r\,d\theta dr
\end{aligned}$$

Therefore, we have determined that

$$A_{0n}\bar{A}_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi} f(r,\theta)J_0(r\lambda_{0n})r\,d\theta dr \tag{2}$$

$$A_{mn}\bar{A}_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \tag{3}$$

$$A_{mn}\bar{B}_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \tag{4}$$

Applying the other initial condition yields the following:

$$\begin{aligned}
g(r,\theta) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\lambda_{mn}J_m(r\lambda_{mn})B_{mn}\left(\cos(m\theta)\bar{A}_{mn}+\sin(m\theta)\bar{B}_{mn}\right) \\
&= \sum_{n=1}^{\infty} J_0(r\lambda_{0n})c\lambda_{mn}B_{0n}\bar{A}_{0n} \\
&+ \sum_{m=1}^{\infty}\left[\left(\sum_{n=1}^{\infty} J_m(r\lambda_{mn})c\lambda_{mn}B_{mn}\bar{A}_{mn}\right)\cos(m\theta) + \left(\sum_{n=1}^{\infty} J_m(r\lambda_{mn})c\lambda_{mn}B_{mn}\bar{B}_{mn}\right)\sin(m\theta)\right]
\end{aligned}$$

Notice that the only difference from what we just looked at in Eq. (1) is that $A_{mn} \longrightarrow c\lambda_{mn}B_{mn}$, and $f \longrightarrow g$. So we can make these substitutions in Eqs. (2)–4), and we arrive at:

$$\begin{aligned}
B_{0n}\bar{A}_{0n} &= \frac{1}{c\lambda_{0n}\pi a^2 J_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi} g(r,\theta)J_0(r\lambda_{0n})r\,d\theta dr \\
B_{mn}\bar{A}_{mn} &= \frac{2}{c\lambda_{mn}\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\cos(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \\
B_{mn}\bar{B}_{mn} &= \frac{2}{c\lambda_{mn}\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\sin(m\theta)J_m(r\lambda_{mn})r\,d\theta dr
\end{aligned}$$

Our solution to the vibrating drumhead can therefore be written as the following:

$$\begin{aligned}
u(r,\theta,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})\big(A_{mn}\cos(c\lambda_{mn}t)+B_{mn}\sin(c\lambda_{mn}t)\big)\big(\bar{A}_{mn}\cos(m\theta)+\bar{B}_{mn}\sin(m\theta)\big) \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})\big(A_{mn}\bar{A}_{mn}\cos(m\theta)+A_{mn}\bar{B}_{mn}\sin(m\theta)\big)\cos(c\lambda_{mn}t)
\end{aligned}$$

$$\begin{aligned}
&+ \quad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})\big(B_{mn}\bar{A}_{mn}\cos(m\theta) + B_{mn}\bar{B}_{mn}\sin(m\theta)\big)\sin(c\lambda_{mn}t) \\
u(r,\theta,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})\big(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\big)\cos(c\lambda_{mn}t) \\
&+ \quad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\lambda_{mn})\big(a^*_{mn}\cos(m\theta) + b^*_{mn}\sin(m\theta)\big)\sin(c\lambda_{mn}t)
\end{aligned}$$

where we have redefined the constants, and using $\alpha_{mn} = a\lambda_{mn}$, we get

$$\begin{aligned}
a_{0n} &= \frac{1}{\pi a^2 J_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi} f(r,\theta)J_0(r\lambda_{0n})r\,d\theta dr \\
a_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \\
b_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \\
a^*_{0n} &= \frac{1}{c\alpha_{0n}\pi a J_1^2(\alpha_{0n})}\int_0^a\int_0^{2\pi} g(r,\theta)J_0(r\lambda_{0k})r\,d\theta dr \\
a^*_{mn} &= \frac{2}{c\alpha_{mn}\pi a J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\cos(m\theta)J_m(r\lambda_{mn})r\,d\theta dr \\
b^*_{mn} &= \frac{2}{c\alpha_{mn}\pi a J_{m+1}^2(\alpha_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\sin(m\theta)J_m(r\lambda_{mn})r\,d\theta dr
\end{aligned}$$