4452 Mathematical Modeling Lecture 10: Linear Approximation, Eigenvalues, Stability

Introduction

We have seen that many of the problems which are encountered in dynamical system modeling are nonlinear in structure. Nonlinear equations can have solutions with qualitative behaviour which is very different globally than a linear equation.

Locally, if the function is differentiable, it can be approximated by a linear function. Therefore, we should look at stability in terms of linear functions, with the knowledge that what we learn can be applied locally around any points of interest.

Solving systems of linear first order differential equations reduces to solving eigenvalue problems, so this analysis has a deep connection with linear algebra. The stability of a system around a point is determined by classifying the eigenvalues of the associated linear system about that point.

These techniques provide the same information as you could extract from a vector field plot, but are more useful in higher dimension when the visualization of a vector field becomes difficult.

Linear Approximation

Consider the nonlinear autonomous system of differential equations

$$\begin{aligned} x_1' = f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1x_2 \\ x_2' = f_2(x_1, x_2) &= x_1 + x_2 - 4x_1x_2. \end{aligned} \tag{1}$$

It is nonlinear because of the terms x_1x_2. Remember, we want to linearize in the phase space variables, not in the time variable. This system can be linearized about the point (x_1^0, x_2^0) by using the fact that

$$\begin{aligned} f_1(x_1, x_2) &\sim f_1(x_1^0, x_2^0) + \frac{\partial f_1(x_1^0, x_2^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f_1(x_1^0, x_2^0)}{\partial x_2}(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim -5x_1^0 + 2x_2^0 + x_1^0x_2^0 + (-5 + x_2^0)(x_1 - x_1^0) + (2 + x_1^0)(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim -5x_1 + 2x_2, \quad x_1 \sim 0, x_2 \sim 0 \end{aligned}$$

where in the last step we choose the point $(x_1^0, x_2^0) = (0, 0)$, since this is one of the fixed points of the system (remember the fixed points are found by solving $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0$ for x_1, x_2). To determine the stability about the other fixed point $(7/19, 7/9)$ you would need to first linearize about that point.

Similarly, we find

$$\begin{aligned} f_2(x_1, x_2) &\sim f_2(x_1^0, x_2^0) + \frac{\partial f_2(x_1^0, x_2^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f_2(x_1^0, x_2^0)}{\partial x_2}(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim x_1^0 + x_2^0 - 4x_1^0x_2^0 + (1 - 4x_2^0)(x_1 - x_1^0) + (1 - 4x_1^0)(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim x_1 + x_2, \quad x_1 \sim 0, x_2 \sim 0. \end{aligned}$$

And so the nonlinear dynamical system in Eq. (1) can be approximated by the linear system about the point $(0, 0)$ as

$$\begin{aligned} x_1' &= -5x_1 + 2x_2 \\ x_2' &= x_1 + x_2, \end{aligned} \tag{2}$$

and about the point $(7/19, 7/9)$ as

$$\begin{aligned} x_1' &= \frac{1}{171}(-49 - 722x_1 + 405x_2) \\ x_2' &= \frac{1}{171}(196 - 361x_1 - 81x_2). \end{aligned} \tag{3}$$

Both of these equations are linear in x_1, x_2. We can proceed to perform the eigenvalue stability analysis, the theory of which we turn our attention to now.

EigenSystems

Typically the ideas of a linear algebra *eigensystem* are introduced to solve systems of linear differential equations. You may need to brush up on your matrix multiplication to understand what follows. Although eigenvectors are not central to the stability discussion, they are an important aspect of the eigensystem and easily understood at a basic level. Bold face quantities indicate matrices and vectors.

The equation $\mathbf{Ax} = \mathbf{y}$ transforms a vector $\mathbf{x}$ into a vector $\mathbf{y}$. What if $\mathbf{y}$ is a multiple of $\mathbf{x}$? Then the vectors are transformed into multiples of themselves:

$$\mathbf{Ax} = \lambda\mathbf{x} \longrightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$\mathbf{I}$ is the identity matrix. This will have nonzero solutions $\mathbf{x}$ if and only if λ are chosen to satisfy:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

λ are the eigenvalues of the matrix, for each λ there is a corresponding eigenvector, $\mathbf{x} = \xi$ This is the essence of an eigensystem.

Example Find the eigenvalues and eigenvectors of the matrix:

$$\mathbf{A} = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix}$$

First, find the eigenvalues:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 7-\lambda & -4 \\ 5 & -2-\lambda \end{vmatrix} = (7-\lambda)(-2-\lambda) + 20 = (\lambda - 3)(\lambda - 2) = 0$$

So the eigenvalues are $\lambda^{(1)} = 2$, $\lambda^{(2)} = 3$. We solve for each eigenvalue in turn:

$\underline{\lambda^{(1)} = 2}$:
Solve the following for $\xi^{(1)}$:

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \lambda^{(1)}\mathbf{I})\xi^{(1)} \\ &= \begin{pmatrix} 7-\lambda^{(1)} & -4 \\ 5 & -2-\lambda^{(1)} \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} 5 & -4 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 5\xi_1^{(1)} - 4\xi_2^{(1)} \\ 5\xi_1^{(1)} - 4\xi_2^{(1)} \end{pmatrix} \end{aligned}$$

We get the two equations:

$$\begin{aligned} 5\xi_1^{(1)} - 4\xi_2^{(1)} &= 0 \\ 5\xi_1^{(1)} - 4\xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. We have one equation in the two unknowns $\xi_1^{(1)}$ and $\xi_2^{(1)}$. Therefore:

$$\begin{aligned} \xi_1^{(1)} &= \text{arbitrary } = c \neq 0 \\ \xi_2^{(1)} &= \frac{5}{4}\xi_1^{(1)} = \frac{5}{4}c \end{aligned}$$

So the eigenvector associated with the eigenvalue $\lambda^{(1)} = 2$ is

$$\xi^{(1)} = c \begin{pmatrix} 1 \\ 5/4 \end{pmatrix} \text{ or } \xi^{(1)} = \begin{pmatrix} 1 \\ 5/4 \end{pmatrix} \text{ or } \xi^{(1)} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$\underline{\lambda^{(2)} = 3}$:
Solve the following for $\xi^{(2)}$:

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \lambda^{(2)}\mathbf{I})\xi^{(2)} \\ &= \begin{pmatrix} 7-\lambda^{(2)} & -4 \\ 5 & -2-\lambda^{(2)} \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 4\xi_1^{(2)} - 4\xi_2^{(2)} \\ 5\xi_1^{(2)} - 5\xi_2^{(2)} \end{pmatrix} \end{aligned}$$

We get the two equations:

$$4\xi_1^{(2)} - 4\xi_2^{(2)} = 0$$

$$5\xi_1^{(2)} - 5\xi_2^{(2)} = 0$$

These are the same equation. We have one equation in the two unknowns $\xi_1^{(2)}$ and $\xi_2^{(2)}$. Therefore:

$$\begin{aligned} \xi_1^{(2)} &= \text{arbitrary } = c \neq 0 \\ \xi_2^{(2)} &= \xi_1^{(2)} = c \end{aligned}$$

So the eigenvector associated with the eigenvalue $\lambda^{(2)} = 3$ is

$$\xi^{(2)} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Summary:

$$\xi^{(1)} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \text{ is eigenvector associated with the eigenvalue } \lambda^{(1)} = 2.$$
$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is eigenvector associated with the eigenvalue } \lambda^{(2)} = 3.$$

This can all be solved for in *Mathematica* using the following commands:

```
matrix = {{7, -4}, {5, -2}}
MatrixForm[matrix]
Eigenvalues[matrix]
Eigenvectors[matrix]
```

Geometric Interpretation of Eigenvalues and Eigenvectors in 2–space

Under the transformation $\mathbf{A}$, the vector $\mathbf{x}$ is transformed into the new vector $\mathbf{y}$ (Fig. 1).

$$\begin{aligned} \mathbf{A}x &= \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -12 \\ -4 \end{pmatrix} = \mathbf{y} \end{aligned}$$

Under the transformation $\mathbf{A}$, the vector $\mathbf{A}\xi^{(1)}$ retains the same orientation as $\xi^{(1)}$–however, it will stretch by an amount $\lambda^{(1)}$ (Fig 2).

- Eigenvectors are determined up to a nonzero constant multiple.
- An n by n matrix will have n eigenvalues (some eigenvalues may repeat).
- each eigenvalue has at least one associated eigenvector, and each eigenvalue of multiplicity m may have q linearly independent eigenvectors where $1 \leq q \leq m$.

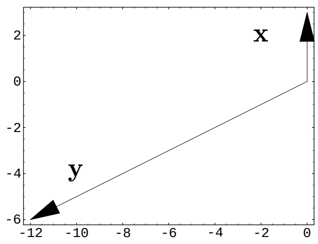
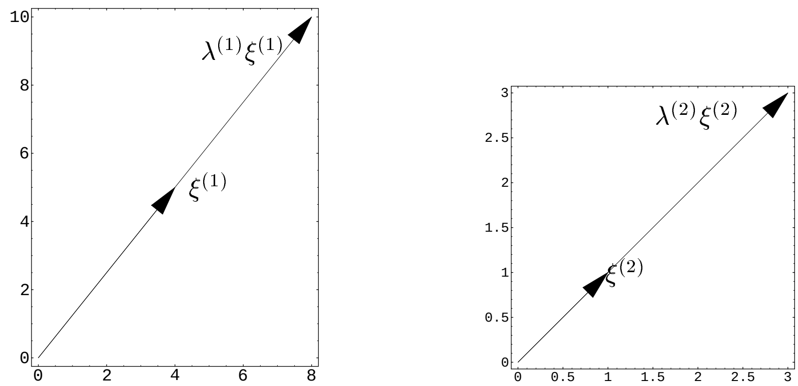


Figure 1: How a general vector transforms under matrix multiplication.



$\xi^{(1)} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\lambda^{(1)} = 2$ $\qquad$ $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda^{(2)} = 3$

$$\mathbf{A}\xi^{(1)} = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix} \quad \mathbf{A}\xi^{(2)} = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Figure 2: How eigenvectors of a matrix transform under matrix multiplication.

Relation of Eigenvalues to Dynamical Systems

Once we have a linear system of differential equations, we can proceed to solve using the eigenvalue method. This assumes that the coefficients in the differential equation are constants.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

$\mathbf{A}$ is a real valued $n \times n$ matrix, although it could in practice be complex valued. Assume a solution of the system which looks like $\mathbf{x} = \xi e^{\lambda t}$ exists, where the ξ and λ are constants which have to be determined. ξ is an n vector, and λ is a scalar. Note that here ξ is NOT a constant of integration! It is a constant vector multiple which is part of the solution.

Substitute into the differential equation:

$$\begin{aligned} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:} \quad & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0: \quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \end{aligned}$$

so solving systems of constant coefficient linear differential equations reduces to finding the eigenvalues λ and eigenvectors ξ of the matrix $\mathbf{A}$!

The solutions are given by $\mathbf{x} = \xi e^{\lambda t}$. Notice that the dynamical behaviour of the solution is determined by the eigenvalues of the matrix, λ, in the $e^{\lambda}t$ part. Spiral points and centers occur when λ is complex valued, and you can use Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$ to get real valued solutions and determine the behaviour. I am including all of the possibilities in Table 1, for completeness sake.

Eigenvalues	Type	Stability
$r_1 > r_2 > 0$	node	unstable
$r_1 < r_2 < 0$	node	asymptotically stable
$r_2 < 0 < r_1$	saddle point	unstable
$r_1 = r_2 > 0$	proper or improper node	unstable
$r_1 = r_2 < 0$	proper or improper node	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$		
$\lambda > 0$	spiral point	unstable
$\lambda < 0$	spiral point	asymptotically stable
$\lambda = 0$	center	stable

Table 1: Stability and instability properties of an $n = 2$ linear system of a differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the eigenvalues of $\mathbf{A}$ are r_1, r_2. Asymptotically stable means that solutions are moving towards the point, but will only reach it after infinite time has passed.

Example Returning to our example, we had linearized and found that about the point $(0, 0)$ Eq. (1) could be approximated by

$$\begin{aligned} x_1' &= -5x_1 + 2x_2 \\ x_2' &= x_1 + x_2. \end{aligned}$$

We are now able to discuss the stability of this point based on the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues are -5.3 and 1.3. This means that the solution will be a decaying exponential in one direction, and a growing exponential in another directions (the directions are related to the eigenvectors that correspond to the eigenvalues, see Fig. 3). This means that the point $(0, 0)$ is an unstable saddle point. This is not surprising because this is the behaviour we already observed with the vector field.

The stability about the point $(7/19, 7/9)$ requires a moment of thought, in that the eigenvalue analysis assumed we had a form that looked like $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which we don't quite have yet.

$$\begin{aligned} x_1' &= \frac{1}{171}(-49 - 722x_1 + 405x_2) \\ x_2' &= \frac{1}{171}(196 - 361x_1 - 81x_2). \end{aligned}$$

What about those constant terms? Well, we should realize that we could rescale the variables to get rid of the constants, and that would not change the behaviour of the solution, it would just shift things to a new location. So we can just, in this case, look at the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} -722 & 405 \\ -361 & -81 \end{pmatrix}$$

which are complex, $-401.5 \pm 208.53i$. Since they are complex, we will get an oscillation (sine and cosine). Since the real part is negative, we will get the solution spiraling in towards the point $(7/19, 7/9)$. So this is a stable spiral point, as we had guessed from the vector field.

I want to reiterate that these methods become extremely important when the dimensionality goes up, and graphical visualization becomes difficult.

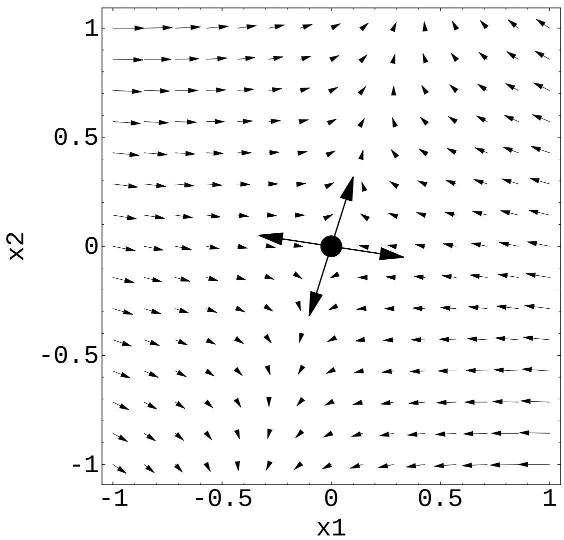


Figure 3: The eigenvectors and the direction of motion around a fixed point. This is for the fixed point $(0,0)$, and the vectors are multiples of the the eigenvectors $\langle -0.99, 0.15\rangle$ (these would be associated with the stable direction, and eigenvalue -5.32) and $\langle -0.30, -0.95\rangle$ (these would be associated with the unstable direction, and eigenvalue 1.32). The motion in those direction is only valid in a small region near the fixed point.

References

[1] W.E. Boyce and R.C. Diprima, *Elementary Differential Equations and Boundary Value Problems* (7th ed.) (New York: John Wiley & Sons, Inc. 2001).