4452 Mathematical Modeling Lecture 2: Automobile Manufacture

Example An automobile manufacturer makes a profit of $1,500 on the sale of a certain model. It is estimated that for every $100 of rebate, sales increase by 15%.

1. What amount of rebate will maximize profit?

Ask the Question

The variables and constants of this problem are, with the units:

Notation	Explanation	Units	Constant or Variable
pm	Profit on the sale of a model of car	$/car	$1500/car
r	Rebate	$/car	Variable
rsf	Rebate scale factor	$/car	$100/car
rin	Sales increase based on rebate	%, no unit	0.15
n	Total number of cars sold <u>when there is no rebate</u>	car	Constant, not given
P	Total Profit	$	Variable

We are assuming:

1. For every $100 in rebate, sales increase by 15%.
2. $r > 0$ or we will be *increasing* the price of the car by offering a rebate.
3. n, the number of cars sold, is an integer.

We want to find the dollar amount of rebate that will maximize the profits to the dealer.

Select the Modeling Approach

We will model this problem as a one dimensional optimization. Once we have the total profit as a function of rebate, we will find the maximum profit by the following procedure:

1. take the derivative of the profit with respect to rebate,
2. set this derivative equal to zero, and
3. solve the resulting equation for the rebate.

This gives us the optimal rebate which will maximize the profit to the dealer. We can check that this profit is a maximum by using the second derivative test, or simply noting that our function is a parabola opening down, so it must have a maximum and not a minimum.

Formulate the Model

We wish to maximize the profit to the dealer. The profit is given by the product of the number of cars sold times the price per car.

I am not 100% sure what the dealer means when they say *It is estimated that for every $100 of rebate, sales increase by 15%.* I would ask the dealer to explain this, but obviously I can't! I will assume that what is meant is this:

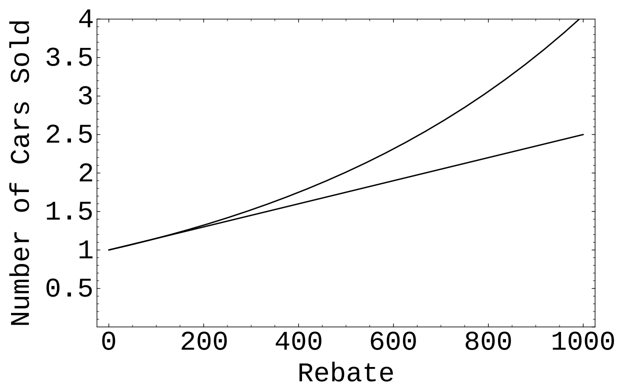
$$\text{number cars sold} = n(1 + 0.15)^{\text{rebate}/100}.$$

This statement says that every time we increase the rebate by $100/car, we increase the sales by 15%.

Aside: when I first solved this, I used:

$$\text{number cars sold} = n(1 + \frac{\text{rebate}}{100}\,0.15),$$

which says something different. It never hurts to check things out, so let's graph these both as a function of rebate and see what we get:



We can see that my first, linear, idea may be useful if there are other constraints that force the rebate to be small. And a linear function is certainly simpler than an exponential! However, let's go with the nonlinear function for number of cars sold, since I think that is more in line with what is meant by our dealer.

The price per car is given by the original price minus the rebate,

$$\text{price per car} = 1500 - \text{rebate}.$$

The profit to the dealer is given by

$$\text{profit to dealer} = n(1 + 0.15)^{\text{rebate}/100}(1500 - \text{rebate}).$$

Since we shall later be doing a sensitivity analysis, let's write this equation in terms of all the constants which we have defined earlier:

$$\text{profit to dealer} = n(1 + rin)^{r/rsf}(pm - r).$$

where the rebate scale factor is rsf =\$100/car; the profit for dealer when no rebate is offered is \$1500/car; and the amount that sales increase based on the rebate of rsf is $rin = 0.15$.

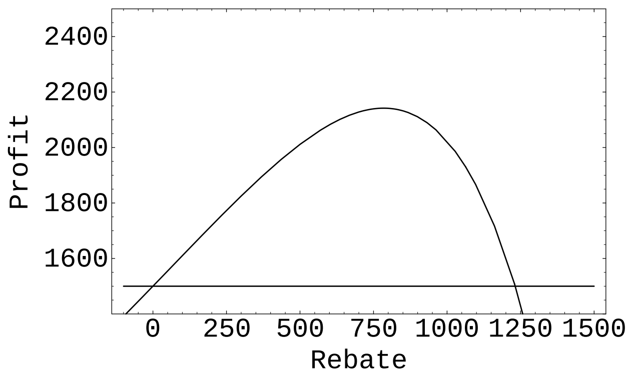
The number of cars sold with no rebate offered, n, is not needed to solve this problem (since we are working with percentages). So we can set $n = 1$, and proceed.

The profit depends only on the rebate, and can be written as

$$P(r) = (1 + rin)^{r/rsf}(pm - r) \qquad \text{or} \qquad P(r) = (1 + 0.15)^{r/100}(1500 - r)$$

Unit check: [car] ([no unit] + [no unit]) $^{[\$/\text{car}]\ /\ [\$/\text{car}]}$ ([\$/car] - [\$/car]) = [\$]
(Remember that n had units of [car], and those units still remain!) Units check out!

To get an idea if our model is correct, let's plot it.



This looks good. It appears there will be a maximum profit for a rebate around \$800/car. We can also see that it looks like there will be a drop in profits if we increase the rebate too much (higher than about \$1250/car), and we see that there is a drop in profits if we have a negative rebate; this situation would mean we are *increasing* the price of cars and losing customers. You can see this directly from the equations.

The *Mathematica* commands used in this section were:

```
pm = 1500;
rin = 15/100;
rsf = 100;
p[r_, n_] = n*(1 + rin)^(r/rsf)(pm - r)
Plot[{p[r, 1], 1500}, {r, -100.00, 1500.0}, Frame -> True,
  PlotRange -> {1400, 2500}, FrameLabel -> {"Rebate", "Profit", "", ""}]
```

Solve the Model

Now, we wish to find the rebate that maximizes the profit. We do that by finding the derivative of the profit, setting it equal to zero, and solving for the rebate. We can see that what we have found is a global maximum since the profit function we have is a parabolic type function opening down, and the only extrema we will find by this method will be the maximum. Alternately, we could use the second derivative test to verify we have a maximum.

The command to do this in *Mathematica* is `Roots[D[p[r, 1] == 0, r], r]`.

The second derivative test is easy to use to verify that a maximum is found when the rebate offered is $784.50. Also, there is no need to work with the symbolic answer, which contains logarithms and is cumbersome (we want an answer in dollars and cents). The dealer's profits per car sold will be $2141.83/car.

Answer the Question

The dealer would maximize their profits if they offered a rebate of $784.50/car.

Sensitivity Analysis

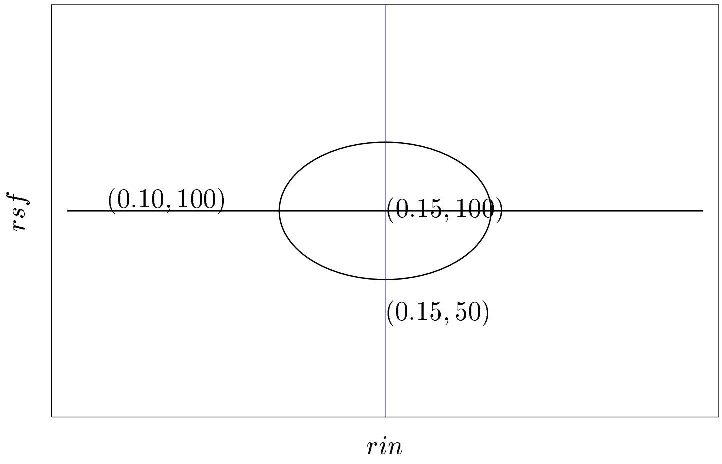
We now ask the question: How fragile is our solution? To answer this, we return to the formulation of the model with all the constants left unspecified:

$$P(r) = (1 + rin)^{r/rsf}(pm - r).$$

There can be a great many things to check in a sensitivity analysis; the trick is to check the right things! We can assume that the profit per car sold with no rebate offered, pm, is fairly accurate, since the dealer would have accurate information to get this number from. We shall not, therefore, perform a sensitivity analysis of our solution relative to this number.

The two other variables in our model are rin and rsf. They represent the increased percentage in sales based on a dollar value of rebate offered per car. These are guesses as to the future behaviour of customers, and exactly the sort of thing that sensitivity analysis should be performed on!

The quantities rin and rsf are related. It is possible that an increase of 15% in sales occurs for a rebate of $50, or maybe $150. Another way of saying the same thing is that $100/car rebate generates only a %10 increase in sales.



We would hope that the actual values of the parameters are close to what we were provided with, say within the circle on our graph.

If we analyze the sensitivity of the optimal rebate with respect to rin, the 15% increase in sales for \$100/car rebate assumption, we are moving across the horizontal line in the graph.

If we analyze the sensitivity of the optimal rebate with respect to rsf, the \$100/car rebate generating 15% increase in sales assumption, we are moving along the vertical line on the graph.

To analyze both at the same time. . . can be difficult to visualize.

To analyze the sensitivity of the optimal rebate with respect to the 15% increase in sales assumption, we need to calculate the optimal rebate with rin left unspecified.

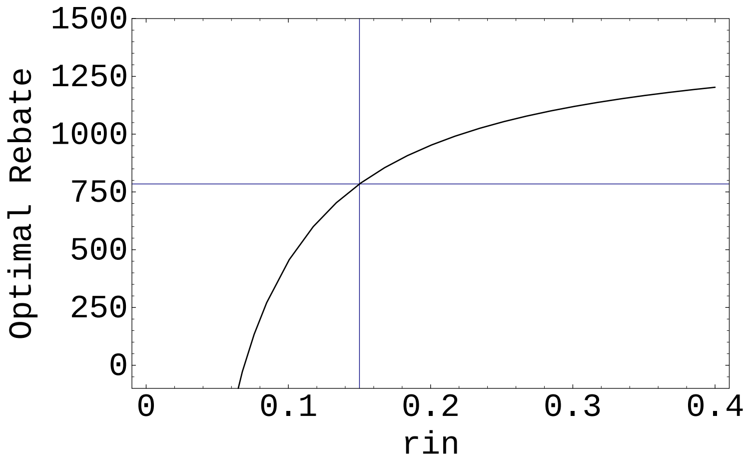
$$P(r) = (1 + rin)^{r/100}(1500 - r).$$

This is far more difficult to do with the exponential function we choose than if we had used a chosen the linear function. We may find it difficult at this point to get answers! However, *Mathematica* makes our lives easier. In *Mathematica*, this is done via:

```
pm = 1500;
Clear[rin];
rsf = 100;
p[r_, n_] = n*(1 + rin)^(r/rsf)(pm - r)
D[profit[r, 1], r]
sol = Solve[Evaluate[D[p[r, 1], r]] == 0, r]
or = r /. Last[sol]
```

And we find the optimal rebate as a function of rin to be

$$r = -\frac{100 - 1500\ln(1 + rin)}{\ln(1 + rin)}.$$



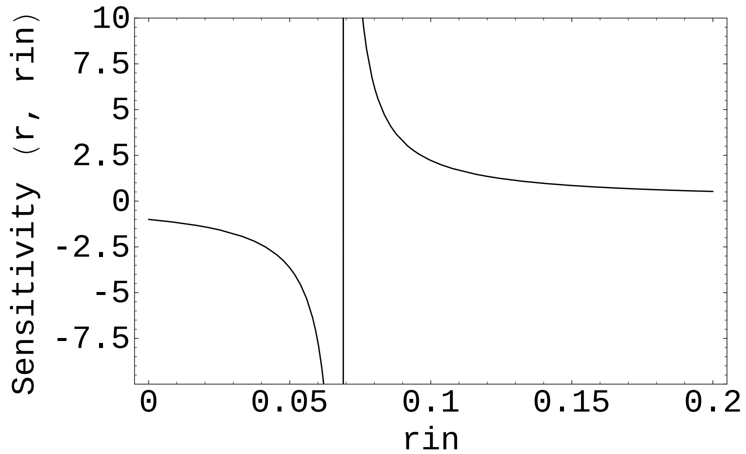
Notice from our graph that the optimal rebate becomes negative for some percentage of new sales, rin. This does not make sense, since it means that we would be increasing the price of the car (negative rebate), and still getting a percentage increase in buyers! Since we know this will not happen, we can restrict rin to have a domain that keeps the optimal rebate positive.

```
Solve[or == 0, rin]
```

And we see that rin must be greater than $-1 + e^{1/15}$.

The sensitivity is given by

$$S(r, rin) = \frac{dr}{d(rin)} \cdot \frac{rin}{r} = \frac{rin}{(1 + rin)\ln(1 + rin)(-1 + 15\ln(1 + rin))}$$



We are working around $rin = 0.15$. $S(r, 0.15) = 0.85 \sim 17/20$, which tells us that a 20% increase in rin results in a 17% increase in the optimal rebate. This seems to be a good, robust result.

If we had estimated rin in our problem to be around 7%, then our results would be fragile, since $S(r, 0.07) = 65$, which is a large sensitivity (1% increase produces a 65% increase).

To analyze the sensitivity of the optimal rebate with respect to rsf, we need to calculate the optimal rebate with rsf left unspecified:

$$P(r) = (1 + 0.15)^{r/rsf}(1500 - r).$$

This is difficult to do, even in *Mathematica*! You could get some plots by plotting the implicit function that is defined in terms of r and rsf.

As noted before, a rebate which is too large for the given percentage increase in sales will cause a reduction in profits. For the parameters we were provided with, this occurs for a rebate of \$1232.00/car. This is found by solving the equation $P(r) = 1500$ for r.

Again, I had trouble getting *Mathematica* to solve this equation. Instead, I found the answer graphically to the nearest dollar. Graphical estimations of quantities can be very helpful in your analysis!