

4452 Mathematical Modeling Lecture 4: Lagrange Multipliers

Lagrange multipliers are a high powered mathematical technique to find the maximum and minimum of multidimensional functions when there are constraints added to the problem. The technique is simple, and easy to implement on a computer. It is used extensively in physics, where I have seen it used to determine the Bose-Einstien statistical distribution obeyed by bosons and equilibrium distributions of gases.

For a function of two variables, the constraint can be thought of as a curve in the xy -plane which the solution *must* lie on. A solution which does not lie on this curve has to be discarded.

Example Say we want to find the maximum of a function,

$$z = f(x, y) = (x - y - 1/4)e^{-x^2 - y^2},$$

subject to the constraint that

$$g(x, y) = (x - 1/2)^2 + (y - 1/3)^2 = 1.$$

If we plot the constraint curve $g(x, y) = 1$ with the level curves of the function, $f(x, y) = c$, we get something that looks like this:

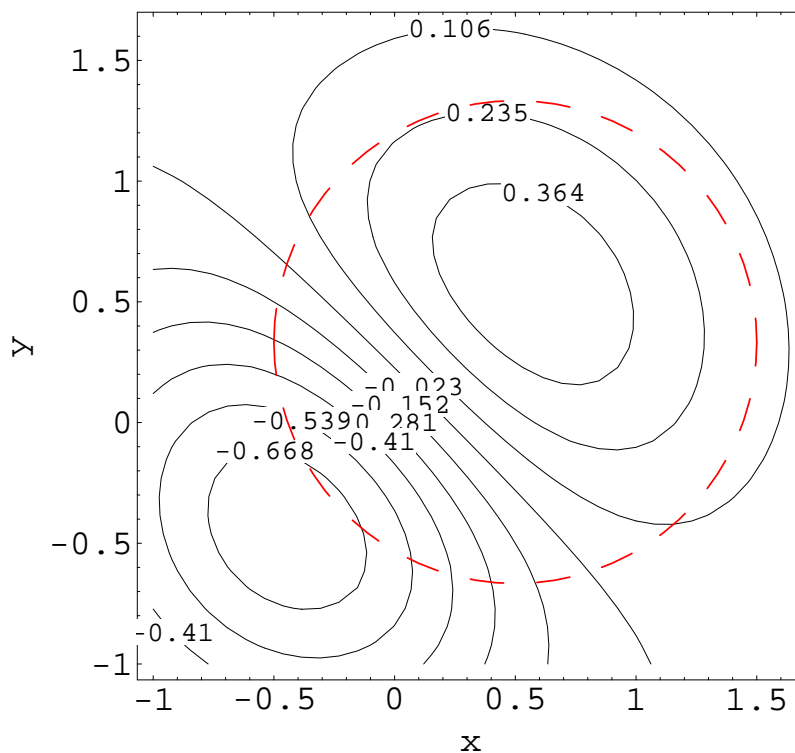


Figure 1: The constraint curve $g(x, y) = 1$ is the dashed line in red, the level curves for the function $f(x, y)$ are the solid curves in black.

We solved for the global maximum of the function f in Lecture 2, and we found that the maximum occurred at the point $(0.566, 0.566)$. However, we can see that this point does *not* lie on the constraint curve, and therefore does not solve the present problem.

Where do you think the maximum of f subject to the constraint g occurs? I think it will be around $(0.1, 1.25)$, since that is where the curve g seems to approach the level curve with largest value (0.235) .

Notice that at this point the constraint curve g and the level curve of the function f appear to be very similar, in the sense that *they have the same slope, or common tangent line*. This is not a coincidence! This is at the heart of the idea of Lagrange multipliers. The concept of Lagrange multipliers will extend to higher dimensions, but this low dimension example makes visualization possible.

The slope of an implicit function can be found by taking the derivative, as we do here for the implicitly defined constraint function:

$$\begin{aligned}
 g(x, y) &= k \\
 \frac{d}{dx}[g(x, y) &= k] \\
 \frac{d}{dx}[g(x, y)] &= 0 \\
 g_x(x, y) \frac{dx}{dx} + g_y(x, y) \frac{dy}{dx} &= 0 \\
 g_x(x, y) + g_y(x, y) \frac{dy}{dx} &= 0 \\
 \frac{dy}{dx} &= -\frac{g_x(x, y)}{g_y(x, y)}, \text{ provided } g_y(x, y) \neq 0.
 \end{aligned}$$

A similar argument can be applied to the level curve, $f(x, y) = c$, of the function we are trying to optimize.

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We want these two slopes to be equal, and we can guarantee that by having

$$f_x(x, y) = \lambda g_x(x, y) \text{ and } f_y(x, y) = \lambda g_y(x, y),$$

where λ is a scalar multiple and is called the Lagrange multiplier. The Lagrange multiplier contains the idea that the two curves can have the same slope at a point, but have very different curvature at that point (Fig. 2).

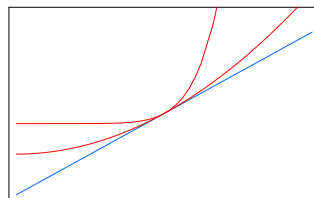


Figure 2: Two curves with the same tangent line at a point but different curvature at that point

Using vector notation and the gradient, $\nabla f(x, y) = (f_x(x, y), f_y(x, y))$, we can write the concept of Lagrange Multipliers as

Lagrange Multipliers To find the maximum and minimum values of $z = f(x, y)$ subject to the constraint $g(x, y) = k$, you need to:

1. Find all the values x, y, λ such that

$$\nabla f(x, y) = \nabla g(x, y), \text{ and } g(x, y) = k.$$

2. Evaluate f at all the points found in step 1. The largest of these is the maximum value of f ; the smallest is the minimum value of f .

You should note that the max and min can occur at more than one point.

Let's finish off the example we had been looking at, where we wanted to find the maximum of $z = f(x, y) = (x - y - 1/4)e^{-x^2 - y^2}$, subject to the constraint that $g(x, y) = (x - 1/2)^2 + (y - 1/3)^2 = 1$.

To solve analytically:

$$\begin{aligned} f(x, y) &= (x - y - 1/4)e^{-x^2 - y^2} \\ f_x(x, y) &= \frac{1}{2}(2 + x - 4x^2 - 4xy)e^{-x^2 - y^2} \\ f_y(x, y) &= \frac{1}{2}(2 + y - 4y^2 - 4xy)e^{-x^2 - y^2} \\ g(x, y) &= (x - 1/2)^2 + (y - 1/3)^2 \\ g_x(x, y) &= 2x - 1 \\ g_y(x, y) &= 2y - 2/3 \end{aligned}$$

$$\text{Solve: } \begin{cases} \frac{1}{2}(2 + x - 4x^2 - 4xy)e^{-x^2 - y^2} = \lambda(2x - 1) \\ \frac{1}{2}(2 + y - 4y^2 - 4xy)e^{-x^2 - y^2} = \lambda(2y - \frac{2}{3}) \\ (x - \frac{1}{2})^2 + (y - \frac{1}{3})^2 = 1 \end{cases} \text{ for } (x, y, \lambda)$$

Wow! This looks difficult (and it is). I used *Mathematica* to solve this system, and even then I had to get tricky (just asking it to solve the three equations simultaneously took a long time and I tired of waiting).

```
Eliminate[{D[f[x, y], x] == LM D[g[x, y], x], D[f[x, y], y] == LM D[g[x, y], y],
  g[x, y] == 1}, LM]
Solve[%, {x, y}];
N[%]
```

This is what *Mathematica* returned (I have removed the small complex part).

Point (a, b)	Functional Value $f(a, b)$	Conclusion
$(-0.262242, -0.313959)$	-0.699	Minimum
$(1.48504, 0.161035)$	0.150	
$(1.47089, 0.572875)$	0.148	
$(0.107593, 1.25313)$	0.228	
		Maximum

Remember that our guess was $(x, y) = (0.1, 1.25)$? The lesson we should take away is sometimes a graphical solution is very valuable, and easier to implement than an analytic method! But the best thing is to understand what you are doing so you can use those graphical tricks.

The result is shown in Fig. 3, with level curves and constraint. The black dots mark the maximum and minimum of the function along the constraint. These plots of level curves (or level sets, as the text calls them) and constraint curves can be very enlightening, and help you explain your results succinctly.

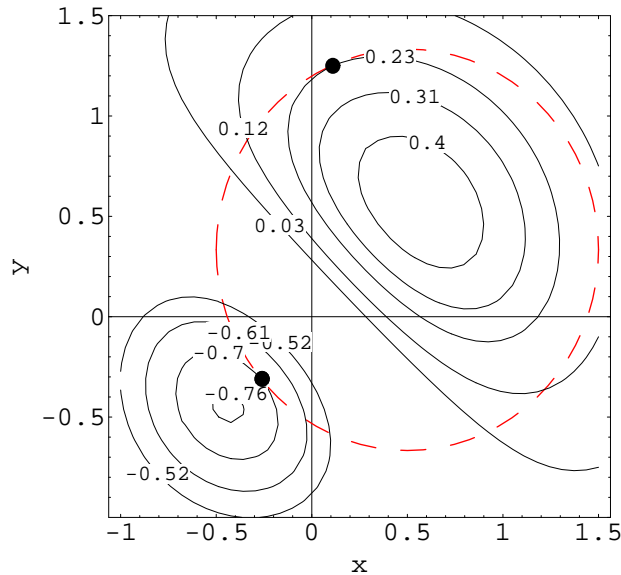


Figure 3: The result showing the max and min with the level curves and constraint curve.

Higher Dimension To maximize the function $w = f(x, y, z)$ subject to the constraints $g_1(x, y, z) = k_1$ and $g_2(x, y, z) = k_2$, you solve the system

$$\begin{aligned}\nabla f(x, y, z) &= \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z) \\ g_1(x, y, z) &= k_1 \\ g_2(x, y, z) &= k_2\end{aligned}$$

for $x, y, z, \lambda_1, \lambda_2$.

You can extend in a similar fashion to even higher dimension.

Inequality Constraints If the problem to be solved has inequality constraints, then first find any extrema using the techniques for unconstrained optimization, then use the Lagrange multiplier technique for any boundaries that exist. Compare all the values you find to determine the maximum and minimum. If you have boundaries that change definition, for instance a square, you may have to use the Lagrange multiplier technique more than once to determine the extrema on the boundary.

Example Find the maximum and minimum of $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ subject to the constraints $x - z \leq 2$ and $y^2 + t \leq 4$.

Since this is a higher dimension problem, it may be difficult to visualize. We also have an inequality restraint. The inequality restraint means we should first find the extrema as if there were no constraints. If the extrema lies within the constrained region, then we are done. If the extrema does not lie within the constrained region, then it must lie on the boundary. We will then proceed to use Lagrange multipliers to find the extrema.

First, the unconstrained optimization. For this we must solve

$$f_x(x, y, z, t) = 0, f_y(x, y, z, t) = 0, f_z(x, y, z, t) = 0, f_t(x, y, z, t) = 0,$$

for (x, y, z, t) . This yields the solution $(0, 0, 0, 0)$. This is obviously a minimum, since the function $f(x, y, z, t) \geq 0$. The point $(0, 0, 0, 0)$ satisfies the constraints, $x - z \leq 2$, and $y^2 + t \leq 4$, so this will be the minimum of the function subject to the constraints.

Since we found no maximum inside the region, the maximum must occur on the boundary. We will use Lagrange multipliers to find it. We want to solve the system:

$$\begin{aligned} f_x(x, y, z, t) &= \lambda_1(g_1)_x(x, y, z, t) + \lambda_2(g_2)_x(x, y, z, t) \\ f_y(x, y, z, t) &= \lambda_1(g_1)_y(x, y, z, t) + \lambda_2(g_2)_y(x, y, z, t) \\ f_z(x, y, z, t) &= \lambda_1(g_1)_z(x, y, z, t) + \lambda_2(g_2)_z(x, y, z, t) \\ f_t(x, y, z, t) &= \lambda_1(g_1)_t(x, y, z, t) + \lambda_2(g_2)_t(x, y, z, t) \\ g_1(x, y, z, t) &= x - z = 2 \\ g_2(x, y, z, t) &= y^2 + t = 4 \end{aligned}$$

for $(x, y, z, t, \lambda_1, \lambda_2)$. Using *Mathematica* to do this, we find:

Lagrange Multipliers (λ_1, λ_2)	Point (x, y, z, t)	Functional Value $f(x, y, z, t)$	Conclusion
$(2, 8)$	$(1, 0, -1, 4)$	18	Maximum
$(2, 1)$	$(1, -\sqrt{7/2}, -1, 1/2)$	5.75	
$(2, 1)$	$(1, \sqrt{7/2}, -1, 1/2)$	5.75	

Notice that the minimum on the boundary of the constraints occurs at two places, and is larger than the minimum we found inside the constraint region. The maximum of the function f subject to the constraints g_1 and g_2 is 18.

Sensitivity Analysis

Sensitivity analysis for Lagrange Multipliers proceeds in much the same way as before, we take a parameter in the problem, change it to a variable, and proceed to solve.

Example We want to find the sensitivity to the parameter α of the maximum of

$$z = f(x, y) = \left(40 - \frac{x}{100} + \frac{\alpha y}{100}\right)x - \left(45 - \frac{5x}{100}\right)y + \frac{3z}{100},$$

subject to the constraint that

$$g(x, y) = 4x + y + z = 10,$$

You can verify that for $\alpha = 6$, the maximum is 14649.9 which occurs at the point (409.364, -288.116, -1339.340).

Solving this via the Lagrange multiplier method, we find

$$f_x(x, y, z) = 40 - \frac{x}{50} + \frac{y}{20} + \frac{\alpha y}{100}$$

$$f_y(x, y, z) = -45 + \frac{x}{20} + \frac{\alpha x}{100}$$

$$f_z(x, y, z) = \frac{3}{100}$$

$$g_x(x, y, z) = 4$$

$$g_y(x, y, z) = 1$$

$$g_z(x, y, z) = 1$$

$$\text{Solve: } \begin{cases} 40 - \frac{x}{50} + \frac{y}{20} + \frac{\alpha y}{100} = 4\lambda \\ -45 + \frac{x}{20} + \frac{\alpha x}{100} = \lambda \\ \frac{3}{100} = \lambda \\ 4x + y + z = 1 \end{cases} \quad \text{for } (x, y, z, \lambda)$$

Lagrange Multiplier λ	Point (x, y, z)
$-\frac{3}{100}$	$\left(\frac{4503}{5 + \alpha}, \frac{-2(5467 + 1994\alpha)}{(5 + \alpha)^2}, \frac{2(-39438 - 6962\alpha + 5\alpha^2)}{(5 + \alpha)^2} \right)$

If we desire $S(x, \alpha)$, we can get it by differentiating:

$$S(x, \alpha) = \frac{dx}{d\alpha} \cdot \frac{\alpha}{x} = -\frac{\alpha}{5 + \alpha}.$$

Similarly,

$$S(y, \alpha) = \frac{dy}{d\alpha} \cdot \frac{\alpha}{y} = \frac{-2\alpha(482 + 997\alpha)}{(5 + \alpha)(5467 + 1994\alpha)}$$

$$S(z, \alpha) = \frac{dz}{d\alpha} \cdot \frac{\alpha}{z} = \frac{2\alpha(22033 + 3506\alpha)}{(5 + \alpha)(-39438 - 6962\alpha + 5\alpha^2)}$$

$$S(f, \alpha) = \frac{df}{d\alpha} \cdot \frac{\alpha}{f} = \frac{-3002\alpha(5467 + 1994\alpha)}{(5 + \alpha)(23171187 + 5986088\alpha + 10\alpha^2)}$$

The graphs of the sensitivities are contained in Fig. 4. Since we are far away ($\alpha \sim 6$) from all the areas where the sensitivity is large, we can safely say that the maximum is robust with respect to the parameter α being approximately 6.

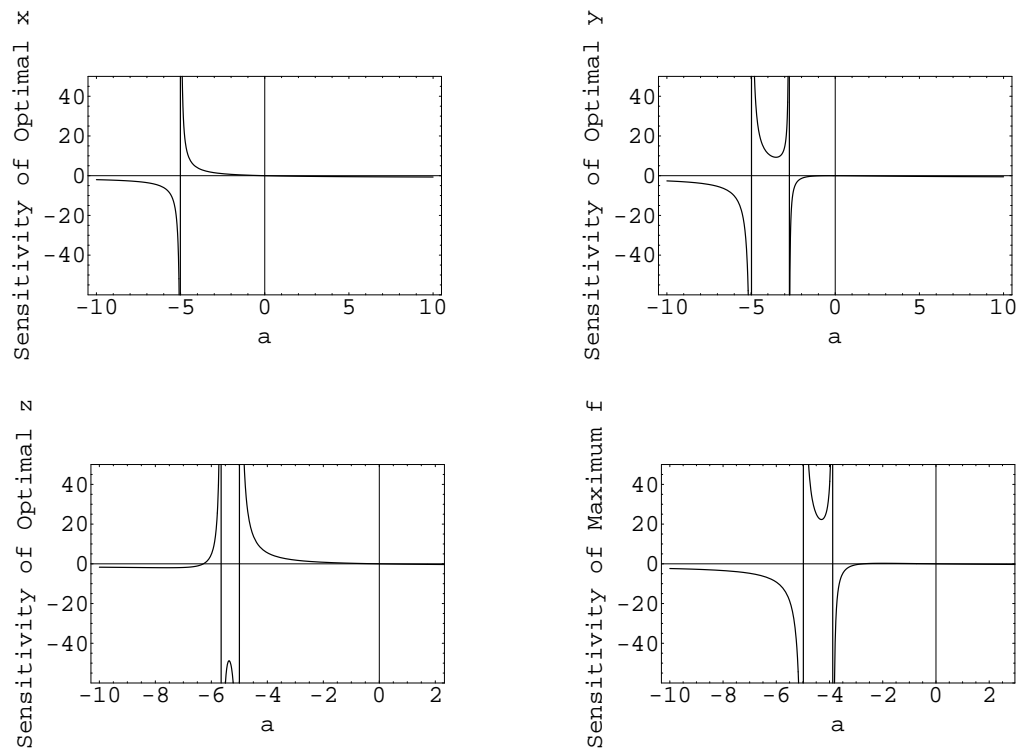


Figure 4: The sensitivity graphs.