4452 Mathematical Modeling Lecture 8: Dynamical Systems

We are now leaving optimization behind, and moving onto the case of dynamical systems. These are systems which change with time, so naturally the modeling will reduce to solving differential equations, since differential equations deal with rates of change. The techniques that are learned in a differential equations class lay the foundation for helping us understand the behaviour of solutions. Let's consider the system of differential equations

$$\begin{aligned} x_1'(t) &= -2x_1(t) + x_2(t) + 2e^{-t} \\ x_2'(t) &= x_1(t) - 2x_2(t) + 3t \end{aligned}$$

which we wish to solve.

The *state space* or *phase plane* is the space of all the dependent variables, here it would be the x_1x_2-plane. If your state space was very large, you could fix some of the variables and look at any x_ix_j-plane you desired.

This is a nonhomogeneous system, since there are the terms e^{-t}, $3t$ which are not multiplied by the quantities x_1 or x_2.

This is also *linear*, since it is linear in the state space variables and their derivatives. Nonlinear differential equations can have behaviour which is much more complicated that that of linear differential equations.

The solution will involve two integrations, so will contain two constants of integration. We will have a doubly infinite family of solutions. To find a particular solution, and create an initial value problem, the initial conditions $x_1(0) = x_1^0, x_2(0) = x_2^0$ would also need to be specified. We typically assume that the system starts at $t = 0$, since t is usually a time.

We can solve differential equations in *Mathematica* using the command DSolve:

```
deq1 = x1'[t] == -2 x1[t] + x2[t] + 2 Exp[-t]
deq2 = x2'[t] == x1[t] - 2 x2[t] + 3 t
DSolve[{deq1, deq2}, {x1[t], x2[t]}, t]
DSolve[{deq1, deq2, x1[0] == 0, x2[0] == 3}, {x1[t], x2[t]}, t]
```

It is very easy to create differential equations which are *extremely* difficult, if not impossible, to solve. This is why the steady state and vector field analysis are so important.

I have found conflicting definitions of steady state and equilibrium in the literature. Our text book uses the following:

An *equilibrium solution* or *fixed point* is one for which the solution remains fixed at a point in the state space. As time progresses, the solution will not change. The system is in a *steady state* when it is at an equilibrium solution.

Some other texts will talk about a steady state solution as being the solution which remains after the transient solution dies off. The transient solution is the part of the solution which becomes negligible after a long period of time. In this case the steady state solution could be an oscillation or growth of the solution in the state space, which is clearly not a fixed point.

By simply being clear about what we mean at any time, any confusion should be minimized.

Let's consider a nonlinear, homogeneous system of differential equations:

$$\begin{aligned} x_1' = f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1x_2 \\ x_2' = f_2(x_1, x_2) &= x_1 + x_2 - 4x_1x_2 \end{aligned} \tag{1}$$

To perform phase plane analysis, we need the system to be of the above form, where there is no explicit dependence on time on the right hand side. Systems with this property are called *autonomous*.

The equilibrium or steady state solutions will satisfy the system of equations

$$\begin{aligned} f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1x_2 = 0, \\ f_2(x_1, x_2) &= x_1 + x_2 - 4x_1x_2 = 0. \end{aligned}$$

The lines in the phase space which these correspond to are called *nullclines*. In general, the line $x_i' = f_i(x_1, x_2) = k$ is called an *isocline* of the function x_i–which is just a fancy way of saying level curve. The $k = 0$ isoclines are called nullclines. The steady state solutions will occur at the points of intersection of the nullclines of the system.

We can solve for the steady state solutions in *Mathematica* using Solve:

```
Solve[{-5x1[t] + 2 x2[t] + x1[t]x2[t] == 0 ,
    x1[t] + x2[t] - 4x2[t]x1[t] == 0}, {x1[t], x2[t]}]
```

and we find the points $(0, 0)$, and $(7/19, 7/9)$. So if the system is at one of these two points, it will remain there. The rates of change will be zero, and so no motion from these two points is possible. Graphically (see Fig. 1), what we have is the intersection of the two implicitly defined curves

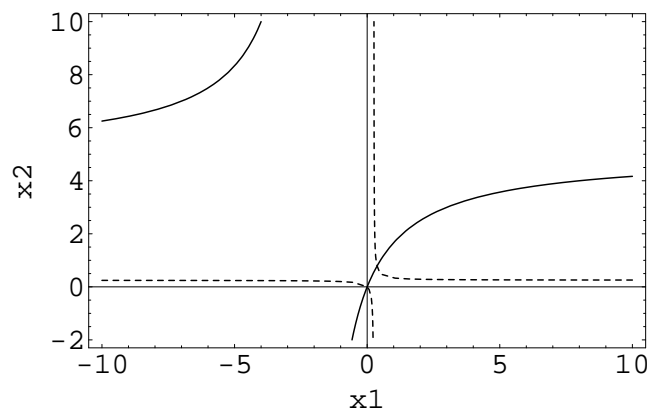$$-5x_1 + 2x_2 + x_1x_2 = 0, \quad x_1 + x_2 - 4x_1x_2 = 0.$$

Figure 1: Plot of the nullclines for the system of differential equations given in Eq. 1. The fixed points for the system are at the intersection of the two nullcines, $(0, 0)$ and $(7/19, 7/9)$. The solid line is the curve $-5x_1 + 2x_2 + x_1x_2 = 0$, and the dashed line is the curve $x_1 + x_2 - 4x_1x_2 = 0$.

From calculus of a single variable, we know that a function changes sign only at zero crossings or discontinuities. This applies to our nullclines, so that we can say that the direction a particle will move changes direction when we cross a nullcline. This will become apparent when we look at vector fields.

The question needs to be asked if the fixed points are stable or unstable equilibrium solutions. A *stable* equilibrium solution means that when the state variables come close to the equilibrium solution they will be drawn to it. An *unstable* equilibrium solution means that when the state variables come close to the equilibrium solution they are pushed away from it. To determine this behaviour, we will need to look at more than just the steady state of the system.

If we can get the solution directly, then we can plot the solution for various initial conditions and determine the stability of the equilibrium solutions. If the system of differential equations is difficult to solve, we may want to *linearize* the differential equation around the fixed points to determine the behaviour near the fixed points. This will be a technique which would be useful once we have read about the eigenvalue method.

Another, graphical means exists to determine the stability. We could use a vector field. Vector fields are useful, since for first order differential equations (differential equations for which the highest derivative is a first derivative) we have everything we need to create the vector field at our fingertips.

The vector field answers the question "If the solution to my dynamical system is at certain points in the state space, which direction will the solution move?" The vector field draws short line segments at a grid of points which indicate how the solution would move if it was at that point. This often provides enough information to make some judgments about the solution.

For the case above, the vector field can be found in *Mathematica* after you have loaded a special package:

```
<<Graphics`PlotField`
field = PlotVectorField[{-5x1[t] + 2 x2[t] + x1[t]x2[t],
     x1[t] + x2[t] - 4x2[t]x1[t]}, {x1[t], -1, 1}, {x2[t], -1, 1},
   Frame -> True, FrameLabel -> {"x1", "x2"}]
```

From Fig. 2 and Fig. 3, we can see that the equilibrium point at $(0,0)$ is semistable, since the solution will move towards it from one side and away from it from another side. However, the solution out at $(7/19, 7/9)$ appears to be stable, as points near it are being drawn into it. Also, we can see that the points are spiraling in to this equilibrium point. Depending on what these differential equations are modeling, we might want to talk about $(7/19, 7/9)$ being a *sink*. This information may well be enough for you to answer the question you have asked.

<u>The Fine Print:</u> The use of "black box" solvers (where the user does not know the method of solution) like NIntegrate or DSolve is fine, in my opinion, as an exploratory or learning tool. When you are beginning to understand a model, you first need some idea of what the model is doing! So for this course, where we are using many different mathematical techniques to do many different things, it is basically alright to not understand the details of the numerical black box you may have used. Our models are to a large extent exploratory in nature. However, that means that it is even more important that you to explain the basic ideas of the model you are using in a manner understandable to a general audience, through things like graphs, vector fields, steady state solutions, and the like. If you go on to pursue modeling as a career, or simply as part of your career, it will become important that you *do* understand the black boxes which you use.
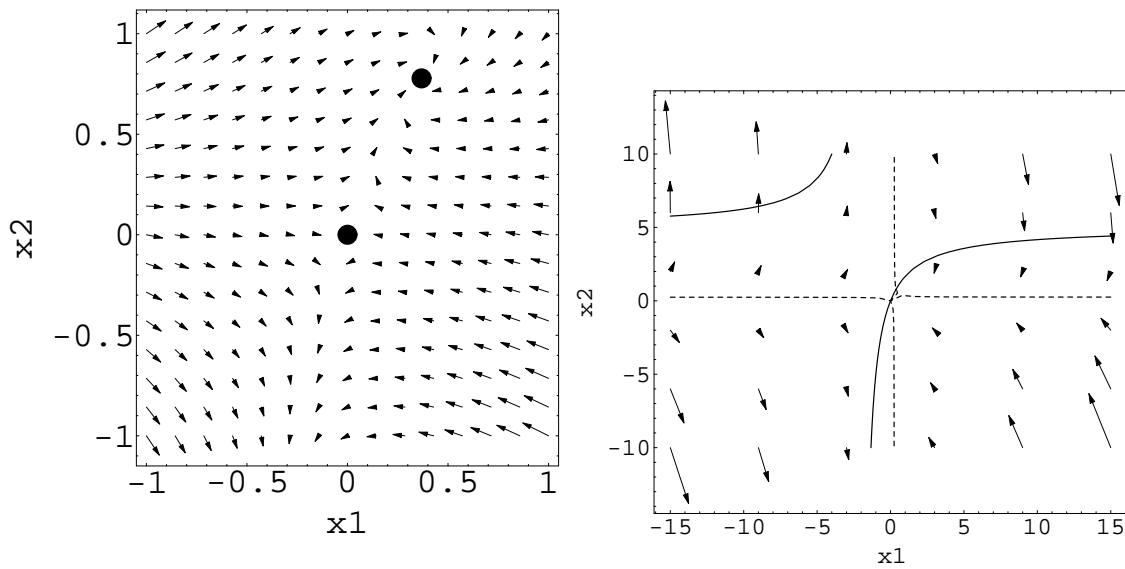
Figure 2: The plot of the vector field to the system of differential equations in Eq. (1). The equilibrium points are indicated by the black dots. The second plot includes the nullclines as well as the vector field. Notice how the direction of motion changes when a nullcline is crossed (this admittedly may be hard to see from this graph!)
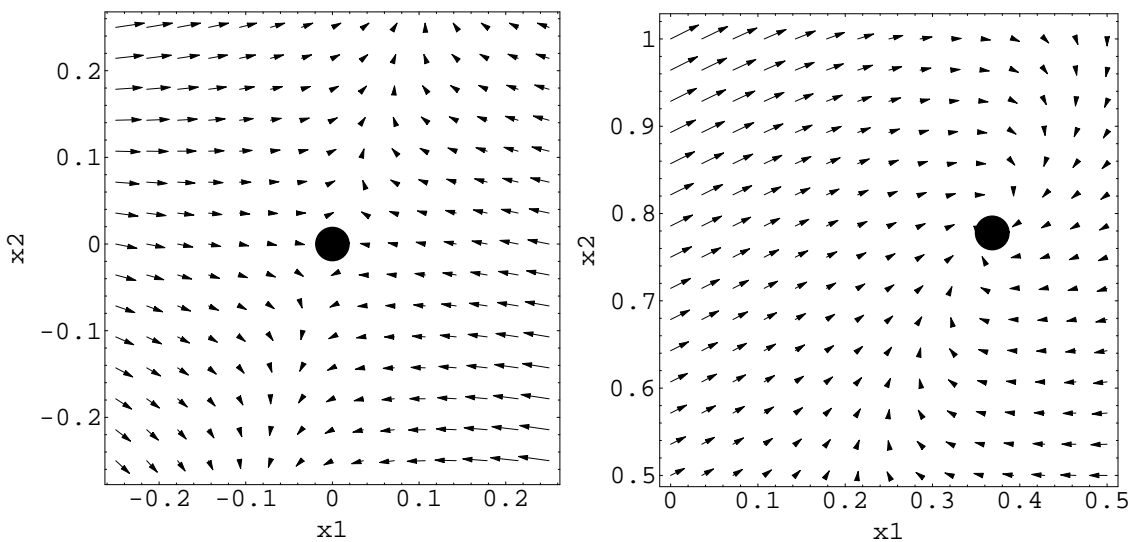

Figure 3: The plot of the vector field to the system of differential equations in Eq. (1). The equilibrium points are indicated by the black dots. The fixed point at $(0,0)$ is seen to be semistable, and the fixed point at $(7/19, 7/9)$ is seen to be stable.

Sensitivity Analysis

A given system which is being modeled will contain initial conditions, which allow you to pick out a particular solution curve from the family of curves that is found when you solve a system of differential equations. As we have seen, the behaviour of solutions depend on where they start, so it is particularly important to do sensitivity analysis on dynamical systems–especially when you have used a "black box" technique to get the solution!

From Fig. 2, we can see that a solution that starts out at $(1, 10^{-2})$ will end up at the stable fixed point. However, it looks like one which starts out at $(1, -10^{-2})$ will trundle off to $(-2, -\infty)$. This is an obvious example of extreme sensitivity to initial conditions, and hopefully not the sort of thing you would run into in practice.

As mentioned before, real world problems may be difficult to solve. You may have to resort to changing a parameter in the model and generating a new set of numerical solutions to compare with the old. You may have to look at a graph of the vector field, and decide how sever changes in the initial conditions will have to be to radically alter the solution, and also, you may have to alter the model, and create new vector fields which will show any changes in the behaviour of the system.

Let's revisit our system.

$$\begin{aligned} x_1' = f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1 x_2 \\ x_2' = f_2(x_1, x_2) &= x_1 + x_2 - 4x_1 x_2 \end{aligned} \tag{2}$$

What if we were interested in how the solution was dependent on the 4? It is the coefficient for a product term, so it is measuring some kind of interaction between x_1 and x_2. An incorrect guess as to the size of this interaction could prove disasterous!

We couldn't solve the original system, and we found our solution by looking at the vector fields. So setting $4 = \alpha$ and proceeding to solve, as much as we want to do that, is not the best way to proceed. Instead, let's create another vector field for a similar system

$$\begin{aligned} x_1' = f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1 x_2 \\ x_2' = f_2(x_1, x_2) &= x_1 + x_2 - 8x_1 x_2 \end{aligned} \tag{3}$$

This will introduce a change only in the nullcline for x_2, so we would expect the position of the stable equilibrium point to change, but not drastically. The stable equilibrium point for the new system is at $(7/39, 7/17)$, so a 100% increase in the value of that interaction parameter moved the stable equilibrium point a distance of

$$\sqrt{\left(\frac{7}{19} - \frac{7}{39}\right)^2 + \left(\frac{7}{9} - \frac{7}{17}\right)^2} = 0.41 \text{ units.}$$

Depending upon the situation, you may wish to comment on a greater shift in x_1 or x_2, or provide more interpretation of this result.

Sensitivity to Ignored Terms

What may be more interesting is that when you set up your model you decided to ignore a nonlinear term in the variable x_2 that contributed to the rate of change of x_1. That would lead to a system like the following

$$
\begin{aligned}
x_1' = f_1(x_1, x_2) &= -5x_1 + 2x_2 + x_1x_2 - 10^{-1}x_2^2 \\
x_2' = f_2(x_1, x_2) &= x_1 + x_2 - 4x_1x_2
\end{aligned} \tag{4}
$$

Since it was ignored originally, you may assume that it has a small contribution. Now how does this affect the solutions of our system?

What you will find is that there is a third equilibrium point which appears, at $(0.25, 21.95)$. It is far away from the other parts of the phase plane we have been considering. However, if the long time behaviour of your model is important, then this effect could become important. If you construct a vector field near the point, you will find that it is a semistable fixed point.

Predator Prey Models (Lotka–Volterra model)

There are two species interacting: a prey species x and a predator species y. For the purposes of the model no other species interact with these two.

In the absence of the predator, the prey exhibits pure exponential growth. There is enough food and resources to allow the prey to grow indefinitely.

In the absence of prey, the predator dies out exponentially. This means that there is other food for the predator than the prey, but not enough to sustain the population. Thus, extinction takes years rather than days.

When there are both predator and prey, the predators kill the prey such that the predator population increases at a rate proportional to the product of the number of predators and the number of prey. The prey population is decreased in a similar manner.

There are lots of assumptions that have gone into this model, not all of which are very realistic. But it is still an excellent place to start, and then improvements could be added later. A detailed look at predator prey relationships might make an excellent final project!

For now, let's convert to a differential equation, and see what we can learn. Let y be the number of predator, and x be the number of prey. Our assumptions above lead us to write:

$$
\begin{aligned}
x' &= ax - bxy \\
y' &= cxy - dy
\end{aligned} \tag{5}
$$

where $a > 0$ is the rate at which prey population increase when there are no predators, $d > 0$ is the rate at which the predators die out when there is no prey, $b > 0$ is the rate at which prey are killed off when there are predators present, and $c > 0$ is the rate and which predators population increases when there are prey present.

The nullclines for this model are

$$\text{Prey nullcline:} \quad ax - bxy = 0 \rightarrow y = a/b, \ x = 0$$
$$\text{Predator nullcline:} \quad cxy - dy = 0 \rightarrow x = d/c, \ y = 0.$$

The fixed points for the system will be at the intersection of the nullclines, which are $(0,0)$ and $(d/c, a/b)$. A complete analysis based on this system could be performed once we begin incorporating the eigenvalue formalism into our solutions. For now, let's simply define our constants and see what we can learn. Let's set

$$a = 1, \ b = 0.4, \ c = 0.5, \ d = 0.1.$$

For these values of the parameters, our nullclines become:

$$\text{Prey nullcline:} \quad y = 5/2, \ x = 0$$
$$\text{Predator nullcline:} \quad \rightarrow x = 1/5, \ y = 0.$$

The vector field for this situation is given in Fig. 4. We can see that there is a fixed point at the intersection of the nullclines. We also see that it is a *stable center*, since our solutions will circle it but not fall into it. What is happening is that the populations oscillate–the predators kill too many prey, resulting in less food and a decrease in their own population, which allows the prey population to increase, then there is enough food so the predator population increases, they kill too many prey, etc.
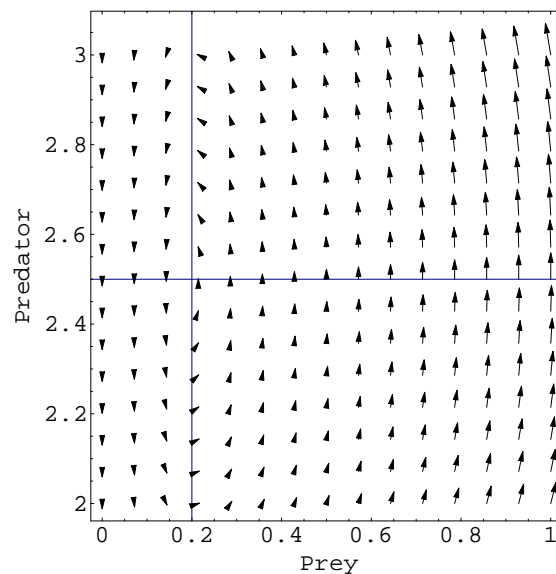

Figure 4: The plot of the vector field to the predator prey model in Eq. (5).

An implicit solution can be found to this equation, but I won't go into it here. There is a lot of information out there on predator prey relationships, and I want to leave this open for someone to do a final project on if they so desire.