Concepts: Algebraic combinations of functions, composition and decomposition of functions.

Algebraic Combinations of Functions

An ambitious way of creating new functions is to combine two or more functions to create a new function.

The most obvious way we can do this is to perform basic algebraic operations on the two functions to create the new one; hence we can add, subtract, multiply or divide functions.

Note that there are two types of algebras in use in this section,

1. the algebra of real numbers, i.e. $4 \times 5 = 20$, $4 - 5 = -1$, $20/10 = 2$, etc.,
2. the algebra of functions, fg, $f - g$, etc.

Algebra of functions

Let f (with domain A) and g (with domain B) be functions. Then the functions $f+g, f-g, fg, f/g$ are defined as:

$(f+g)(x) = f(x) + g(x)$ domain $A \cap B$

$(f-g)(x) = f(x) - g(x)$ domain $A \cap B$

$(fg)(x) = f(x)g(x)$ domain $A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $\left\{x \in A \cap B \,\middle|\, g(x) \neq 0\right\}$

The domains are all the *intersection* (that's what the symbol $\cap$ means) of the domain of f and g, making sure we don't divide by zero.

A Closer Look

- The minus sign in $f - g$ represents *the difference between two functions.*
- The minus sign in $f(x) - g(x)$ represents *the difference between two real numbers.*

The relation that we have that allows us to calculate this quantity is $(f - g)(x) = f(x) - g(x)$, which is easy to remember.

This is a subtle point, but it is always a good idea to understand what the mathematical notation is telling you.

Note: Two functions are equal if they have the same functional definition and the same domain.

Example If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find the functions $f+g, f-g, fg, f/g$ and give their domains.

First, we need to determine the intersection of the domains of f and g, so we need to determine the domains of f and g.
Domain of $f = \sqrt{x}$ is $x \in [0, \infty)$
Domain of $g = \sqrt{4 - x^2}$ is such that $4 - x^2 \geq 0 \rightarrow -2 \leq x \leq 2$ or $x \in [-2, 2]$.
Therefore the intersection of these domains is $x \in [0, 2]$ or $0 \leq x \leq 2$.
And our new functions are defined as:
$(f+g)(x) = \sqrt{x} + \sqrt{4 - x^2}$, $0 \leq x \leq 2$
$(f-g)(x) = \sqrt{x} - \sqrt{4 - x^2}$, $0 \leq x \leq 2$
$(fg)(x) = \sqrt{x}\sqrt{4 - x^2} = \sqrt{4x - x^3}$, $0 \leq x \leq 2$
$(f/g)(x) = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$, $0 \leq x < 2$, where we exclude $x = 2$ since it would lead to division by zero.

Example If $f(x) = \sqrt{x}$, find the function ff and give the domain.

First, we need the domain of f: Domain of $f = \sqrt{x}$ is $x \in [0, \infty)$
Our new function is defined as
$(ff)(x) = \sqrt{x}\sqrt{x} = x,\ 0 \leq x \leq \infty$.
So the basic function $h(x) = x, x \in \mathbb{R}$ is NOT the same function as $(ff)(x) = x,\ 0 \leq x < \infty$ since the domains are different.

Composition of functions

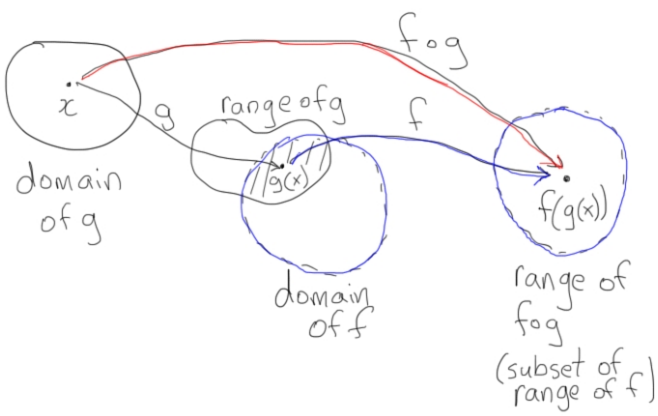
Given two functions f and g, the *composite function* $f \circ g$ (called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

and the domain of $f \circ g$ consists of all x-values in the domain of g that map to $g(x)$ values in the domain of f.

It is important to note that $f \circ g \neq g \circ f$.

Arrow Diagram of Composition



Example Find $(f \circ g \circ h)(x)$ if $f(x) = \dfrac{x}{x+1}$, $g(x) = x^{10}$, $h(x) = x + 3$.

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) \\
&= f(g(x+3)) \\
&= f((x+3)^{10}) \\
&= \frac{(x+3)^{10}}{(x+3)^{10}+1}
\end{aligned}$$

Since the domain and range of h is $x \in \mathbb{R}$, and the domain and range of g is $x \in \mathbb{R}$, there are no restrictions on the domain from the set we are drawing from. The only restriction arises from division by zero, but since $(x+3)^{10}+1 = 0$ has no real valued solutions, the domain of $f \circ g \circ h$ is $x \in \mathbb{R}$.

Composition example If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and give the domains.

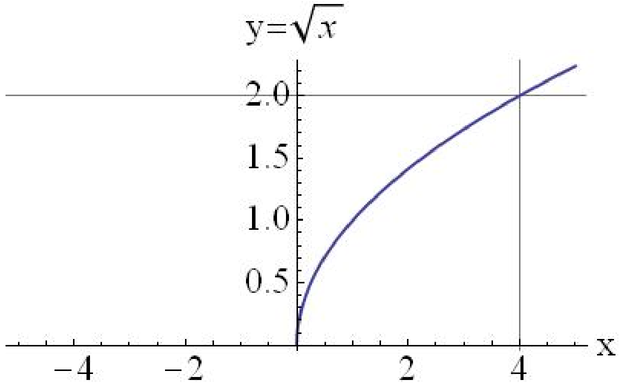
Solution

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{4 - x^2}) \\ &= \sqrt{(\sqrt{4 - x^2})} \\ &= (4 - x^2)^{1/4} \end{aligned}$$

For x to be in the domain of $f \circ g$, we must first find $g(x) = \sqrt{4 - x^2}$, which we can do for $x \in [-2, 2]$.
Then, we take the square root of the result, which we can always do since the range of $g(x) = \sqrt{4 - x^2}$ is the set $[0, \infty)$.
Therefore, the domain of $f \circ g$ is $x \in [-2, 2]$.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= \sqrt{4 - (\sqrt{x})^2} \\ &= \sqrt{4 - x} \end{aligned}$$

For x to be in the domain of $g \circ f$, we must first find $f(x) = \sqrt{x}$, which we can do for $x \in [0, \infty)$. The range of f is $[0, \infty)$.
Then, we must be able to square the result, subtract from 4, and take a square root!
We can do this if we restrict the range of f to be $[0, 2]$.
Here is a sketch to help us find which values in the domain of f give the range in the set $[0, 2]$.



The domain of $g \circ f$ is $[0, 4]$.
Note this is different than if we had just looked at the domain of $h(x) = \sqrt{4 - x}$, which is $x \leq 4$.
The lesson: the domain of compositions cannot be found simply by looking at the final function relation.

Example of Decomposing using composition Given $F(x) = \cos^2(x + 9)$, determine funcions f, g, h so you can write $F(x)$ as a composition $(f \circ g \circ h)(x)$.

Look at how you compute $F(x)$, and build the functions from that:
Add 9: $h(x) = x + 9$
Take cosine: $g(x) = \cos x$
Square: $f(x) = x^2$
You should check that this choice yields $(f \circ g \circ h)(x) = F(x)$.