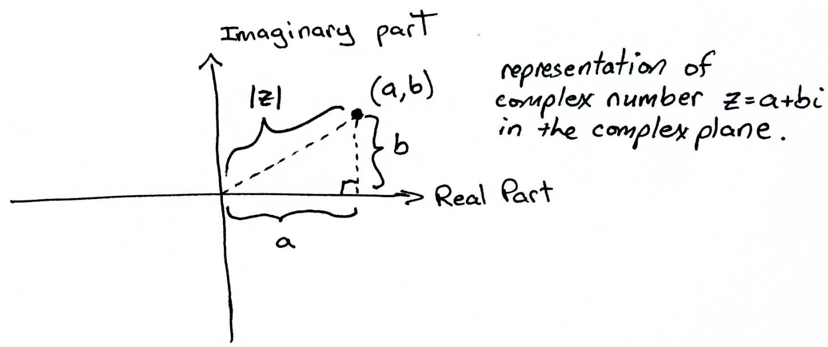
Concepts: basics of complex numbers, Fundamental Theorem of Algebra, polynomials with real coefficients, complex conjugate zeros, polynomials of odd degree.

Complex Numbers

A complex number is typically denoted $z \in \mathbb{C}$. We write the complex number in terms of its real part a and imaginary part b as $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

Two complex numbers are equal if and only if their real and imaginary parts are equal.

A complex number can be represented as a point on the *complex plane*:



Notice that if $a \neq 0, b = 0$, the number a is a real number (as it lies on the real axis). *Therefore, all real numbers are also complex numbers.*

If $a = 0, b \neq 0$, the number bi is called an imaginary number (as it lies on the imaginary axis).

The **complex conjugate** of $z = a + bi$ is defined as $\bar{z} = a - bi$.

The magnitude of a complex number $z = a + bi$ is given by the formula

$$\begin{aligned} |z| &= \sqrt{z\bar{z}} \\ &= \sqrt{(a+bi)(a-bi)} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

Note: $|z|$ reduces to the familiar absolute value if $z \in \mathbb{R}$.

Arithmetic with Complex Numbers

Addition and subtraction: collect real and imaginary parts:

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$$

Multiplication: use the distributive property and the fact $i^2 = -1$ (do not memorize this formula):

$$\begin{aligned} (a+bi)(c+di) &= a(c+di) + bi(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division: multiply the numerator and denominator by the complex conjugate of the denominator (do not memorize this formula):

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i\end{aligned}$$

The Fundamental Theorem of Algebra: A polynomial of degree n has n complex zeros (real and nonreal), some of which may be repeated (a repeated zero will have multiplicity greater than one).

The Fundamental Theorem of Algebra tells us we may always factor $f(x)$, a polynomial function of degree n, as

$$f(x) = a(x-z_1)(x-z_2)\cdots(x-z_{n-1})(x-z_n)$$

where a is the leading coefficient of $f(x)$ and z_i are the complex zeros of $f(x)$. The Fundamental Theorem of Algebra does not tell us *how* to determine this factorization, and proving the Fundamental Theorem of Algebra involves some very advanced mathematics.

Important: A non-real zero z_i of f is *not* an x-intercept of the graph of $f(x)$. For example, $f(x) = x^2+1$ has zeros $z_1 = i$ and $z_2 = -i$ and factorization $f(x) = (x-i)(x+i)$ yet $f(x)$ is a quadratic that opens up with vertex at $(0,1)$ so it does not cross the x-axis and has no x-intercepts.

This is only a bit of weirdness that crops up since we could be working with functions $f(z)$ defined on the complex numbers (with complex coefficients), but that is a *significantly* more complicated topic! We are just dipping our toe into the waters of complex numbers here, and are still firmly rooted in working with real valued functions (with real valued coefficients).

Complex conjugate zeros If f is a polynomial with real coefficients, and $a+bi$ is a zero of f, then $a-bi$ is also a zero of f. That is, *complex valued zeros of a polynomial with real coefficients always occur in complex conjugate pairs.*

Polynomial of odd degree Since a polynomial of odd degree will have end behaviour that goes to positive infinity on one side and negative infinity on the other side (and polynomials are continuous), it must touch the x-axis at least once. This means *all polynomials of odd degree will have at least one real valued zero.* Another way of thinking of this is that since complex zeros must occur in complex conjugate pairs, and there will be an odd number of zeros, at least one of the zeros must be real valued.

Example Find the unique polynomial of degree 4 that has zeros $1-2i$ and $1+i$ that passes through the point $(0,20)$.

Unless we are explicitly told otherwise, we assume the polynomial has real coefficients. Therefore, the complex zeroes will occur in complex conjugate pairs. We can write the polynomial as

$$\begin{aligned}f(x) &= a[x-(1-2i)][x-(1+2i)][x-(1+i)][x-(1-i)]\\ &= a[x^2-x(1-2i)-x(1+2i)+(1-2i)(1+2i)][x^2-x(1-i)-x(1+i)+(1+i)(1-i)]\\ &= a[x^2-x+\cancel{2xi}-x-\cancel{2xi}+1-4i^2][x^2-x+\cancel{xi}-x-\cancel{xi}+1-i^2]\\ &= a[x^2-2x+5][x^2-2x+2]\\ &= a(x^2[x^2-2x+2]-2x[x^2-2x+2]+5[x^2-2x+2])\\ &= a(x^4-2x^3+2x^2-2x^3+4x^2-4x+5x^2-10x+10)\\ &= a(x^4-4x^3+11x^2-14x+10)\end{aligned}$$

Notice we needed to include the leading coefficient, and use the fact that the polynomial passes through $(0,20)$ to determine the value of a. $f(0) = 10a = 20$, so $a = 2$. Therefore, $f(x) = 2x^4-8x^3+22x^2-28x+20$.

Example Given that $1+i$ is a zero of $f(x) = x^4 - 2x^3 - x^2 + 6x - 6$, find all the zeros and write a linear factorization of f.

Since $1+i$ is a zero, so is its complex conjugate $1-i$ (coefficients of f are real valued).

Therefore, $f(x)$ has factors

$$\begin{aligned}(x-(1+i))(x-(1-i)) &= (x^2 - x(1+i) - x(1-i) + (1+i)(1-i)) \\ &= x^2 - x - \cancel{xi} - x + \cancel{xi} + 1 - i^2 \quad \text{use } i^2=-1 \\ &= x^2 - 2x + 2\end{aligned}$$

use long division:

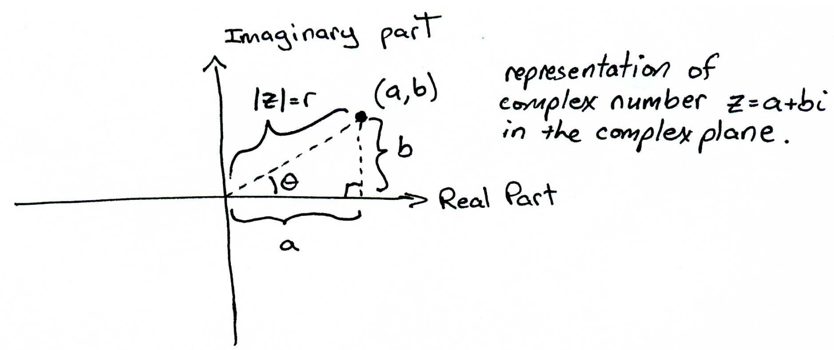
$$\begin{array}{r l} & \quad x^2 - 3 \\ x^2-2x+2 & \overline{\big) x^4 - 2x^3 - x^2 + 6x - 6} \\ & \underline{x^4 - 2x^3 + 2x^2} \\ & \qquad -3x^2 + 6x - 6 \\ & \qquad \underline{-3x^2 + 6x - 6} \\ & \qquad\qquad 0 \end{array}$$

zeros: $x = 1 \pm i, \pm\sqrt{3}$

$$\begin{aligned} \Rightarrow f(x) &= (x^2 - 2x + 2)(x^2 - 3) \\ &= (x^2 - 2x + 2)(x - \sqrt{3})(x + \sqrt{3}) \quad \text{is the irreducible factorization (the quadratic is irreducible over the reals)} \\ &= (x-(1+i))(x-(1-i))(x-\sqrt{3})(x+\sqrt{3}) \quad \text{complete linear factorization}\end{aligned}$$

Aside and Looking Ahead: Connection of Complex Numbers with Trigonometry and Exponentials

Looking ahead, there is a deep and beautiful connection between complex numbers, trigonometry, and exponentials. Looking back at our diagram of the complex plane, we can introduce an angle θ and distance $r = |z|$ from polar coordinates.



This allows us to write

$$\begin{aligned} z &= a + bi \\ &= |z|\cos\theta + |z|\sin\theta i \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

Looking ahead even further, there is a result that can be proven using calculus known as Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$. This means that we can write complex numbers in a variety of ways:

$$z = a + bi = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

These powerful representations can be used to prove trig identities, determine roots of complex numbers, solve linear differential equations, and a host of other things.

In my opinion, Euler's formula is one of the most amazing formulas in mathematics. It kicks ASS!!