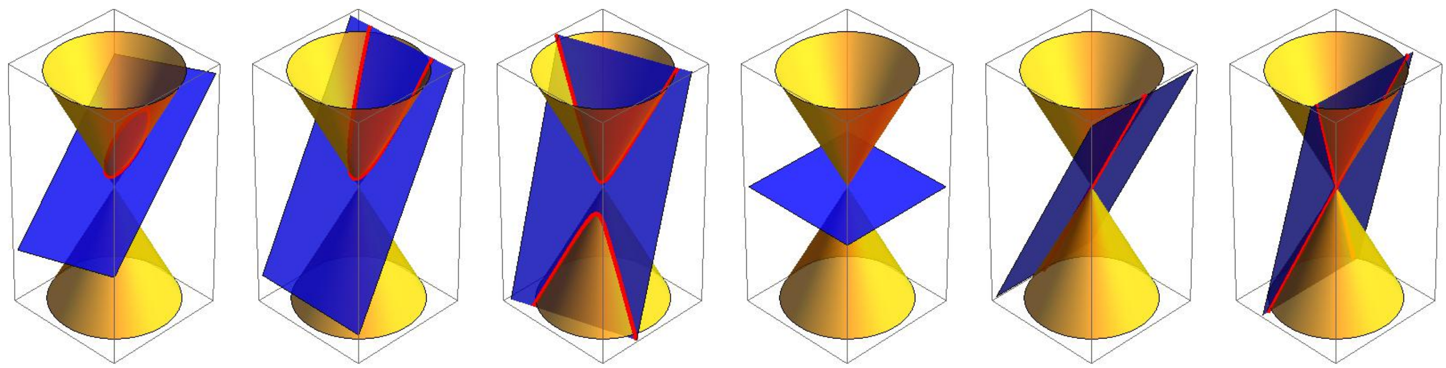
Concepts: Conic Sections, parabolas (focus, directrix, focal axis, focal length, focal width, reflective property, sketching).

What is a Conic Section

If you slice through a cone with a plane, you get a variety of objects in the plane. These are called *conic sections*, which are the red lines in the diagrams below.



Ellipse | Parabola | Hyperbola | Point | Single Line | Intersecting Lines

The latter three cases (point, single line and intersecting line) are degenerate conic sections. The basic conic sections are the parabola, ellipse (including circles), and hyperbolas.

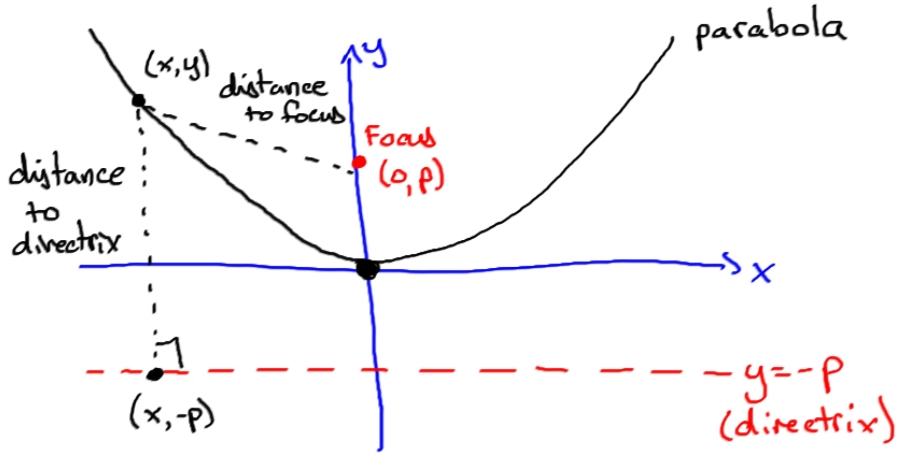
Although there are many interesting properties of the conic section, we will focus on the derivations of the algebraic equations for parabolas, circles, ellipses, hyperbolas, and sketching these by hand. Sketching will be an important skill in future math and science classes.

Definition of a Parabola

I've included the precise definition of a parabola here for completeness, but often people will call $y = x^2$ a parabola, and we've already seen how to sketch this function, as well as $y = ax^2 + bx + c$.

Definition: A parabola is the set of all points in a plane equidistant from a particular line (the directrix) and a particular point (the focus) in the plane.

Let's derive the algebraic equation for a parabola. Without loss of generality, we can assume the focus is $(0, p)$ and the directrix is the line $y = -p$ (without loss of generality just means that any other situation could be transformed into this case).

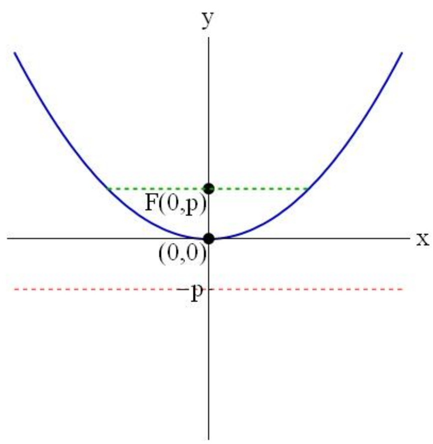


From the definition of parabola, we must have for an arbitrary point (x, y) on the parabola:

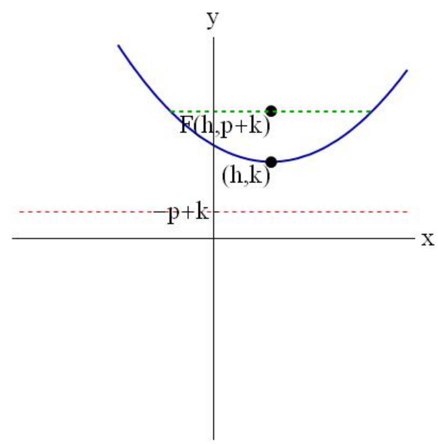
$$\begin{aligned}
\text{distance to focus} &= \text{distance to directrix} \\
\sqrt{x^2 + (y-p)^2} &= \sqrt{(y+p)^2} \\
x^2 + (y-p)^2 &= (y+p)^2 \\
x^2 + y^2 + p^2 - 2yp &= y^2 + p^2 + 2yp \\
x^2 &= 4yp
\end{aligned}$$

The standard form for the equation of a parabola is $x^2 = 4yp$ (opens up) or $y^2 = 4xp$ (opens to right).

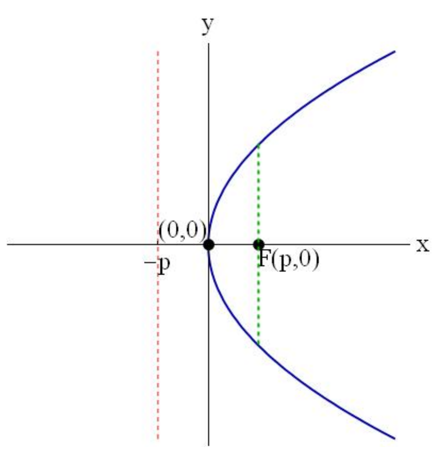
The transformed form which moves the vertex from the origin to (h, k) is $y - k = 4p(x-h)^2$ or $x - h = 4p(y-k)^2$.



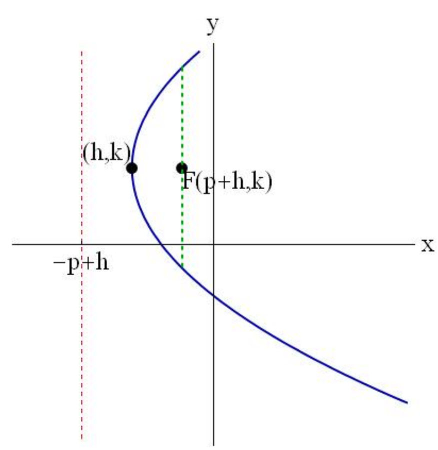
Standard Form: $x^2 = 4yp$



Transformed Form: $(x-h)^2 = 4(y-k)p$



Standard Form: $y^2 = 4xp$



Transformed Form: $(y-k)^2 = 4p(x-h)$

The point F is the focus, and the green dashed line has length $|4p|$ (the focal width). The red dashed line is the directrix.

Parabolic mirrors with this particular geometry can be used to focus light at the focal point, F. Parabaloids (a parabola rotated in three dimensional space about its axis) can be used to treat tumors.

Note on Formulas: There are many formulas associated with conic sections. I recommend knowing how to sketch parabolas, ellipses, and hyperbolas by hand by understanding the basic properties of each. For parabolas, you should definitely know how to determine the locations of the directrix, vertex, and focus.

Example. Sketch $(x+4)^2 = -12(y+1)$.

$(x+4)^2 = -12(y+1)$

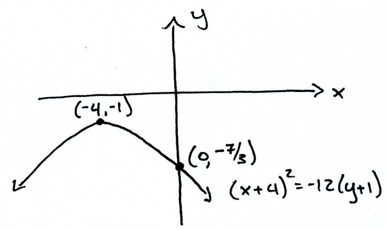
Compare to standard form $(x-h)^2 = 4p(y-k)$

This minus sign means it opens down.

The vertex is at $(h,k) = (-4,-1)$.

Let's get another point. If $x=0$, then

$$4^2 = -12(y+1) \Rightarrow y = -7/3$$



Example. Sketch $y^2 - 2y + 4x - 12 = 0$.

Since this is squared in y but not x, we expect to find a parabola that opens left or right.

$y^2 - 2y + 4x - 12 = 0$

$\underbrace{y^2 - 2y + 1} - 1 + 4x - 12 = 0$

$(y-1)^2 + 4x - 13 = 0$

$(y-1)^2 + 4(x - 13/4) = 0$

$(y-1)^2 = -4(x - 13/4)$

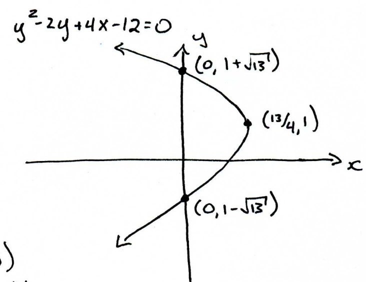
compare to standard form: $(y-k)^2 = 4p(x-h)$

The minus sign means it opens to left, not right.

The vertex is at $(h,k) = (13/4, 1)$.

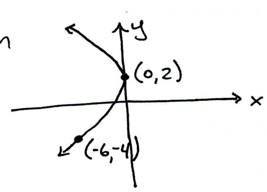
If $x=0$, $(y-1)^2 = 13$

$y = 1 \pm \sqrt{13}$ (just to get more points of interest on the curve).



Example. Write an equation for a parabola that opens to the left, with vertex $(0,2)$ and passes through $(-6,-4)$.

sketch



Standard form for opening to left:

$$(y-k)^2 = -4p(x-h)$$

Vertex is $(0,2) \Rightarrow h=0,\ k=2$.

$$(y-2)^2 = -4p(x)$$

Use $(-6,-4)$ to determine value of p:

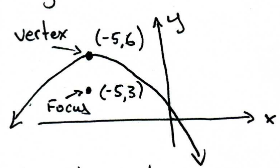
$$(-4-2)^2 = -4p(-6) \Rightarrow p = 3/2$$

$$\Rightarrow (y-2)^2 = -4(3/2)x$$

$$(y-2)^2 = -6x$$

Example. Write an equation for a parabola with focus $(-5, 3)$ and vertex $(-5, 6)$.

Rough sketch:



It must open down.

Standard form: $(y-k)^2 = -4p(x-h)$

Vertex $(-5,6) \Rightarrow h = -5,\ k = 6$

$$(y-6)^2 = -4p(x+5)$$

Distance from vertex to focus is $p = 6 - 3 = 3$.

$$\Rightarrow (y-6)^2 = -12(x+5)$$

check: if $x = -5$, $y = 6$ we get $0 = 0$. Good!