Concepts: Exponential Functions, the base e, logistic functions, properties.

Laws of Exponents To work algebraically with exponential functions, we need to use the laws of exponents. You should memorize these laws.

If x and y are real numbers, and $b > 0$ is real, then

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$

Exponential Functions

A function of the form $f(x) = a \cdot b^x$ is an *exponential function*, where $a \neq 0$ and $b > 0, b \neq 1$ are real numbers. The number b is called the base of the exponential function.

Note: The difference between a monomial and an exponential is where the variable is.
monomial: $f(x) = x^3$ has variable x in base.
exponential: $f(x) = 3^x$ has variable x in the exponent.
constant: $f(x) = 3^\pi$ is a constant function (no x).
more complicated function: $f(x) = x^x$ has an x in the base and in the exponent (you will see these in calculus).

Exponential functions are defined for all real numbers. For some real numbers, it is easy to figure out what the exponential function is. Consider $f(x) = 3^x$, which is an exponential function.

Evaluate at an integer: $f(4) = 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$.

Evaluate at zero: $f(0) = 3^0 = 1$.

Evaluate at negative integer: $f(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81} \sim 0.012345$.

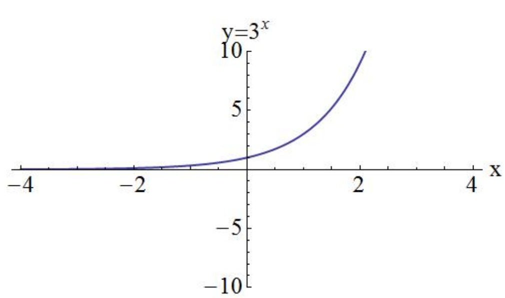
Evaluate at a rational number: $f(-3/2) = 3^{-3/2} = \frac{1}{3^{3/2}} = \frac{1}{\sqrt{27}} \sim 0.19245$.

However, we cannot so easily figure out what an exponential function is when we evaluate it at a irrational number, for example what is $f(\pi) = 3^\pi$.

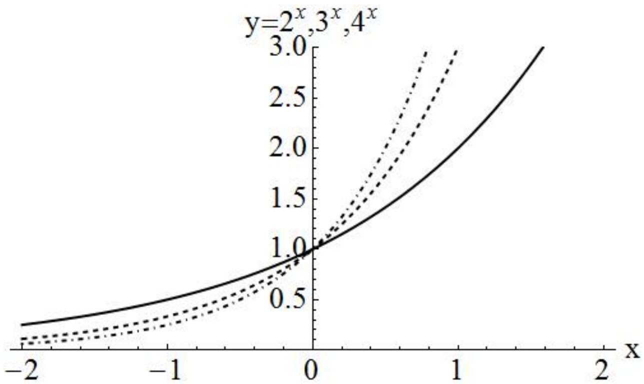
We can determine the value of the number 3^π by a procedure which will get us as close as we want to the number:

$$\begin{aligned}
3^\pi &\sim 3^3 = 27 \\
3^\pi &\sim 3^{31/10} = 30.1353 \\
3^\pi &\sim 3^{314/100} = 31.4891 \\
3^\pi &\sim 3^{314159/100000} = 31.5442 \\
3^\pi &\sim 31.5443
\end{aligned}$$

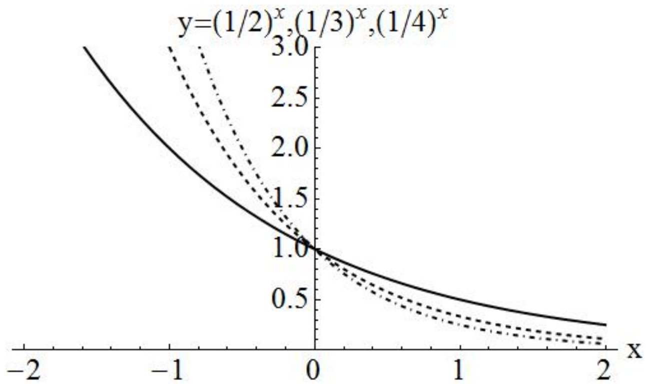
In this way, we can evaluate an exponential at any real value for x. The sketch of the exponential function looks like:



Sketches with different bases:



Solid: $y = 2^x$, Dashed: $y = 3^x$, DotDashed: $y = 4^x$



Solid: $y = (1/2)^x$, Dashed: $y = (1/3)^x$, DotDashed: $y = (1/4)^x$

Growth or Decay (Increasing or Decreasing)

If $a > 0$ and $b > 1$, then $f(x) = a \cdot b^x$ is increasing and called an *exponential growth function* (b is called the *growth factor*).

If $a > 0$ and $0 < b < 1$, then $f(x) = a \cdot b^x$ is decreasing and called an *exponential decay function* (b is called the *decay factor*).

This is also sometimes stated the following way, since for $b > 1$ the quantity $1/b$ will be less than one, so we have exponential decay with $\left(\frac{1}{b}\right)^x = (b^{-1})^x = b^{-x}$, and we can say:

If $a > 0$ and $b > 1$, then $a \cdot b^x$ is increasing, and $a \cdot b^{-x}$ is decreasing.

Example Determine the exponential function that passes through the points $(0, 5)$ and $(2, 16)$.

The general exponential function is $f(x) = a \cdot b^x$. We can use the two points we are given to determine the two constants a and b.

$$f(0) = 5 = a \cdot b^0 = a$$

so $a = 5$.

$$f(4) = 16 = 5 \cdot b^2$$

You might want to review the Properties of Exponents on page 8. We have to use these properties to solve this equation.

$$\begin{aligned} 16 &= 5 \cdot b^2 \\ \frac{16}{5} &= b^2 \\ \left(\frac{16}{5}\right)^{1/2} &= (b^2)^{1/2} \\ \sqrt{\frac{16}{5}} &= b \end{aligned}$$

The exponential function that passes through the two points is $y = f(x) = 5(\sqrt{16/5})^x$. We do not use the negative root for b since for exponential functions the base must be greater than zero.

The base e

There is a preferred base in calculus (you will see why in calculus). It is an irrational number $e = 2.718281828\ldots$. In calculus you will learn why this base is preferred to other bases. In calculus, you will also see where the definition of the number e

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

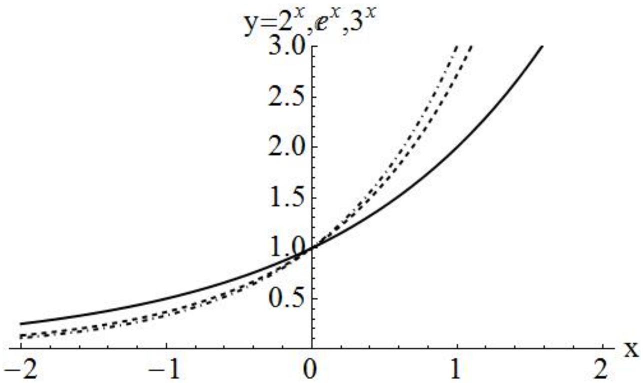
comes from, although it can be motivated using the idea of *continuously compounded interest*.

For now, remember that e is simply an irrational number, and it is the preferred base for the exponential function.

The Natural Exponential Function

$$f(x) = a \cdot e^{kx}$$

If $a > 0$ and $k > 0$, this is an exponential growth function.
If $a > 0$ and $k < 0$, this is an exponential decay function.



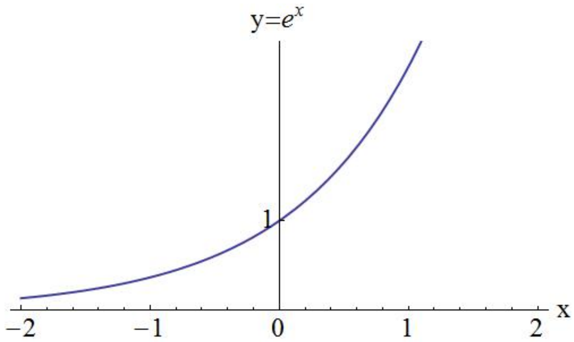
Once we learn about logarithms, we shall see that we can always convert from a general base b to base e using $b^x = e^{kx}$. Logarithms will allow to determine the value of k for a given b, but even without logarithms we can see this is true using the rules of exponents:

$$\begin{aligned} f(x) &= b^x \text{ (rewrite } b = e^k \text{ for some value of } k) \\ &= (e^k)^x \\ &= e^{kx} \end{aligned}$$

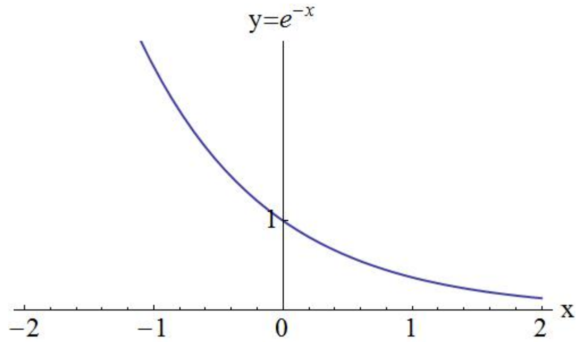
For this reason, I tend to focus on $y = e^{kx}$ for exponential growth and $y = e^{-kx}$ for exponential decay, although we shall see that when modeling with exponentials it is often more beneficial to use a different base.

The exponential function can be transformed using our typical transformation techniques (for example, $y = 2^{x+3}$ is $y = 2^x$ shifted to the left by three units).

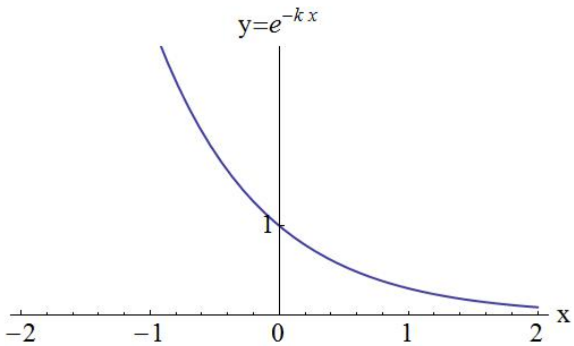
Example Sketch $y = ae^{-kx}$ where $a > 0$ and $k > 0$ are real numbers.



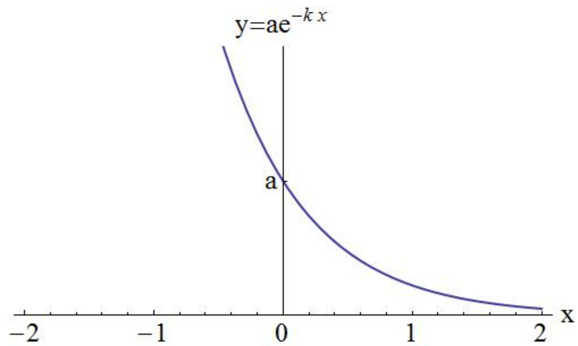
Basic Function $y = f(x) = e^x$



$y = f(-x) = e^{-x}$ reflect about y-axis



$y = f(-kx) = e^{-kx}$ horizontal compression by factor of k



$y = af(-kx) = ae^{-kx}$ vertical stretch by a factor of a

From the sketch we can analyze the behaviour of the function $f(x) = ae^{-kx}$ where $a > 0$ and $k > 0$:
Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Decreasing
End Behaviour: $\lim_{x \to -\infty} ae^{-kx} = \infty$ and $\lim_{x \to \infty} ae^{-kx} = 0$

Logistic Functions

Both logistic and exponential functions are used to model population growth. The problem with exponential growth is that it is unbounded, and at some point the model will no longer predict what is actually happening since the amount of resources available is finite and exponential growth cannot continue on a purely physical level.

If a, b, c, and k are positive constants, and $b < 1$, then a *logistic growth function* can be written as

$$f(x) = \frac{c}{1 + a \cdot b^x} = \frac{c}{1 + a \cdot e^{-kx}}$$

Logistic growth is bounded above and below, so it is a more realistic model of population growth.

Using the sketch of $y = ae^{-kx}$ from the previous example, we can actually construct the sketch of the logistic equation.

First, let's determine the end behaviour of the logistic function:

$$\lim_{x\to\infty} \frac{c}{1+a\cdot e^{-kx}} = \frac{c}{1+a\cdot 0} = c$$
$$\lim_{x\to-\infty} \frac{c}{1+a\cdot e^{-kx}} = \frac{c}{1+a\cdot \infty} = 0$$

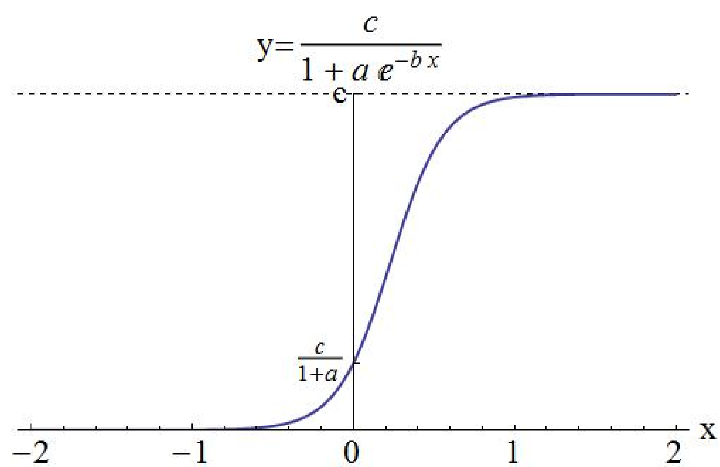
So the logistic function has two horizontal asymptotes, $y = 0$ and $y = c$.

Does the logistic function have any vertical asymptotes? The answer would be yes if $1 + a \cdot e^{-kx} = 0$ since then we would have division by zero. This would happen if $ae^{-kx} = -1$, which never happens since the range of ae^{-kx} is $(0, \infty)$. So the logistic function has no vertical asymptotes, and it is continuous on the domain $x \in (-\infty, \infty)$.

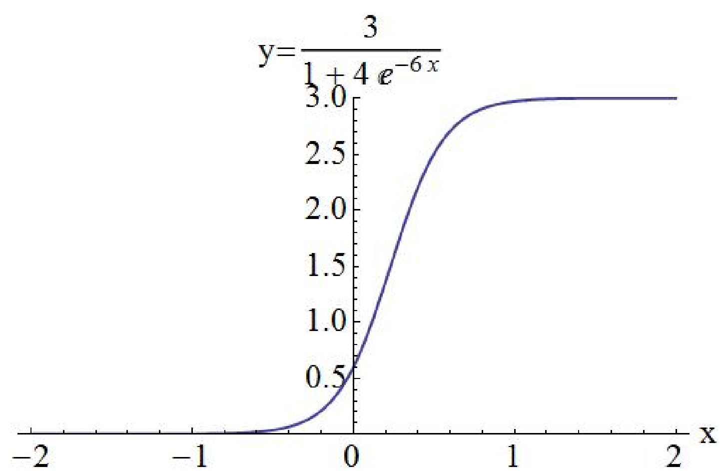
The logistic function has no zeros, since $c \neq 0$.

The y-intercept of the logistic function is $f(0) = \dfrac{c}{1+a}$.

Putting all this information together, we can get a sketch of the logistic function

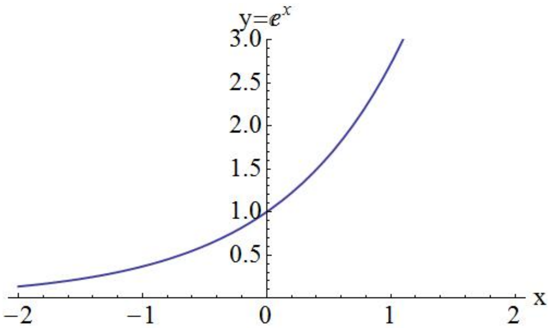


In particular, here is a sketch of $f(x) = \dfrac{3}{1+4e^{-6x}}$:



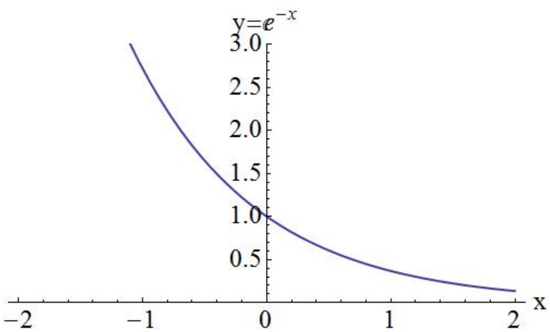
If you continue with math, you will see improvements to the logistic growth that take into account extinction of a species (this is typically done in a differential equations course).

Properties of $y = e^x$



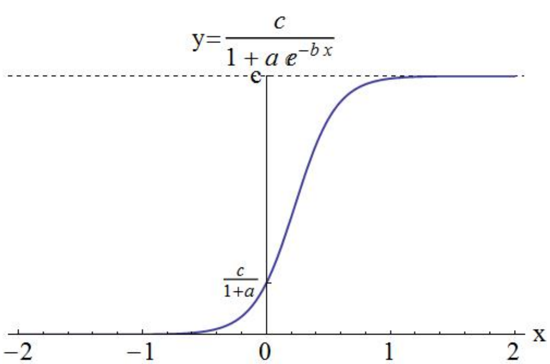
Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Increasing
End Behaviour: $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Properties of $y = e^{-x}$



Domain: $x \in \mathbb{R}$
Range: $y \in (0, \infty)$
Continuous
Neither odd nor even
Bounded below, but not above
No local extrema
Horizontal Asymptote: $y = 0$
No Vertical Asymptotes
Decreasing
End Behaviour: $\lim_{x \to -\infty} e^{-x} = \infty$ and $\lim_{x \to \infty} e^{-x} = 0$

Properties of $y = \dfrac{c}{1 - ae^{-bx}}$



Domain: $x \in \mathbb{R}$
Range: $y \in (0, c)$
Continuous
Neither odd nor even
Bounded above and below
No local extrema
Horizontal Asymptotes: $y = 0$ and $y = c$
No Vertical Asymptotes
Increasing
End Behaviour: $\lim_{x \to -\infty} \dfrac{c}{1 - ae^{-bx}} = 0$ and $\lim_{x \to \infty} \dfrac{c}{1 - ae^{-bx}} = c$