Concepts: Basic Identities, Pythagorean Identities, Cofunction Identities, Even/Odd Identities.

Basic Identities

From the definition of the trig functions:

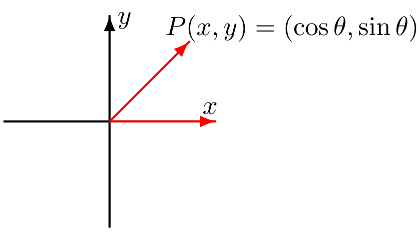
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

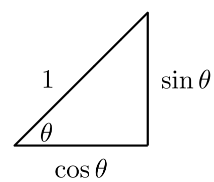
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

Consider a point on the unit circle:



which leads to triangle



Using the Pythagorean theorem, we see that (memorize this one): $\boxed{\cos^2\theta + \sin^2\theta = 1}$

Derive two other identities from the one we have memorized:

Divide by $\cos^2\theta$:

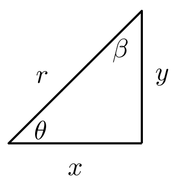
$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \Rightarrow \qquad 1 + \tan^2\theta = \sec^2\theta$$

Divide by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \Rightarrow \qquad \cot^2\theta + 1 = \csc^2\theta$$

Cofunction Identities

Consider the reference triangle:



We have from the reference triangle:

$$\sin\theta = \frac{y}{r} = \cos\beta \qquad \cos\theta = \frac{x}{r} = \sin\beta \qquad \tan\theta = \frac{y}{x} = \cot\beta$$

$$\csc\theta = \frac{r}{y} = \sec\beta \qquad \sec\theta = \frac{r}{x} = \csc\beta \qquad \cot\theta = \frac{x}{y} = \tan\beta$$

The angles must satisfy $\theta + \beta = \frac{\pi}{2}$, $\beta = \frac{\pi}{2} - \theta$. Therefore,

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \qquad \tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) \qquad \sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Even/Odd Identities (from sketches of trig functions)

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Example Use the cofunction and even/odd identities to prove $\cos(\pi - x) = -\cos x$.

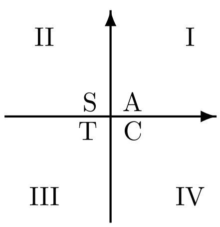
$$\begin{aligned}
\cos(\pi - x) &= \cos\left(\frac{\pi}{2} - \left(x - \frac{\pi}{2}\right)\right) && \text{want to use } \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \\
&= \sin\left(x - \frac{\pi}{2}\right) && \text{want to use } \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \\
&= \sin\left(-\left(\frac{\pi}{2} - x\right)\right) && \text{use } \sin(-\theta) = -\sin\theta \\
&= -\sin\left(\frac{\pi}{2} - x\right) \\
&= -\cos(x)
\end{aligned}$$

Example Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = 0.8$ and $\tan\theta < 0$.

We shall use trig identities rather than reference triangles, or coordinate system, which is how we would have solved this before.

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
\sin^2\theta &= 1 - \cos^2\theta \\
\sin\theta &= \pm\sqrt{1-\cos^2\theta} \\
&= \pm\sqrt{1-(0.8)^6} \\
&= \pm\sqrt{1-0.64} \\
&= \pm\sqrt{0.36} \\
&= \pm 0.6
\end{aligned}$$

We need to figure out the correct sign.



When $\cos\theta > 0 \Rightarrow P$ is in either QI or QIV.
When $\tan\theta < 0 \Rightarrow P$ is in either QII or QIV.

We are in Quadrant IV. In Quadrant IV, $\sin\theta < 0$.

Therefore, $\sin\theta = -0.6$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-0.6}{0.8} = -0.75.$

Example Simply $\dfrac{(\sec y - \tan y)(\sec y + \tan y)}{\sec y}$ to a basic trig function.

$$\begin{aligned}
\frac{(\sec y - \tan y)(\sec y + \tan y)}{\sec y} &= \frac{\sec^2 y - \tan^2 y}{\sec y} && \text{Multiply numerator} \\
&= \frac{1}{\sec y} && \text{use } \sec^2 y - \tan^2 y = 1 \\
&= \cos y
\end{aligned}$$

Example Simplify $\sin x \cos x \tan x \sec x \csc x$ to a basic trig function:

$$\begin{aligned}
\sin x \cos x \tan x \sec x \csc x &= \frac{1}{\cancel{\csc x}}\frac{1}{\cancel{\sec x}}\tan x \cancel{\sec x}\cancel{\csc x} \\
&= \tan x
\end{aligned}$$

Example Write $\dfrac{\tan^2 x}{\sec x + 1}$ as an algebraic expression involving a single trig function:

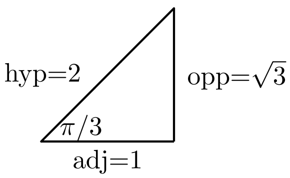
$$\begin{aligned}
\frac{\tan^2 x}{\sec x + 1} &= \frac{\sec^2 x - 1}{\sec x + 1} && \text{use } \sec^2 y - \tan^2 y = 1 \\
&= \frac{(\sec x - 1)(\sec x + 1)}{\sec x + 1} && \text{use difference of squares in numerator } a^2 - b^2 = (a+b)(a-b) \\
&= \sec x - 1
\end{aligned}$$

Example Find all the solutions to the equation $4\cos^2 x - 4\cos x + 1 = 0$.

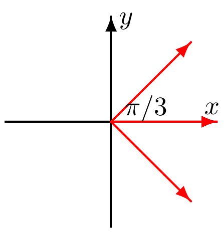
Note this equation is quadratic in $\cos x$. Let $y = \cos x$.

$$\begin{aligned} 4\cos^2 x - 4\cos x + 1 &= 0 \\ 4y^2 - 4y + 1 &= 0 \\ y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{16 - 16}}{8} \\ &= \frac{1}{2} \quad \text{multiplicity 2} \\ 4y^2 - 4y + 1 &= 4\left(y - \frac{1}{2}\right)^2 = 0 \end{aligned}$$

So now we must solve $y = \cos x = 1/2$. This comes from one of our special triangles.



Therefore, $x = \pi/3$. What other angles will be solutions?



We see the other solution is in Quadrant IV, and is $-\pi/3$.
We can also have solutions which are multiples of 2π, so the solution to the original equation is
$x = \pm\frac{\pi}{3} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$.

Example Find all the solutions to the equation $\sqrt{2}\tan x\cos x - \tan x = 0$ in the interval $[0, 2\pi)$.

$$\begin{aligned} \sqrt{2}\tan x\cos x - \tan x &= 0 \\ \tan x\left(\sqrt{2}\cos x - 1\right) &= 0 \qquad \text{factor} \end{aligned}$$

So we have either $\tan x = 0$, or $\sqrt{2}\cos x - 1 = 0$.

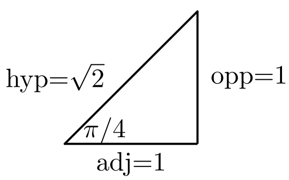
Solve $\tan x = 0$:

$$\begin{aligned} \tan x &= 0 \\ \frac{\sin x}{\cos x} &= 0 \\ \sin x &= 0 \\ x &= 0,\ \pi \end{aligned}$$

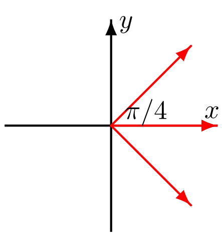
Solve $\sqrt{2}\cos x - 1 = 0$:

$$\begin{aligned} \left(\sqrt{2}\cos x - 1\right) &= 0 \\ \cos x &= \frac{1}{\sqrt{2}} \end{aligned}$$

The angle comes from one of our special triangles:



There are two solutions in $[0, 2\pi)$:



The solutions are $\pi/4$ and $2\pi - \pi/4 = 7\pi/4$.
The solutions to $\sqrt{2}\tan x\cos x - \tan x = 0$ in $[0, 2\pi)$ are $x = 0, \dfrac{\pi}{4}, \pi, \dfrac{7\pi}{4}$.