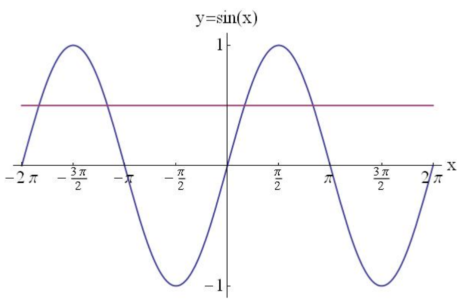
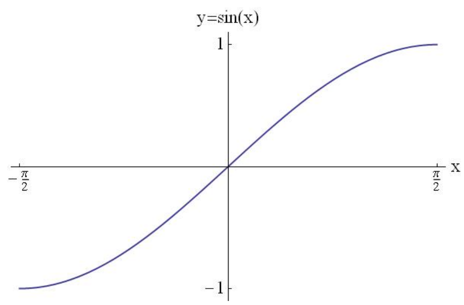
Concepts: Inverse sine, inverse cosine, inverse tangent.

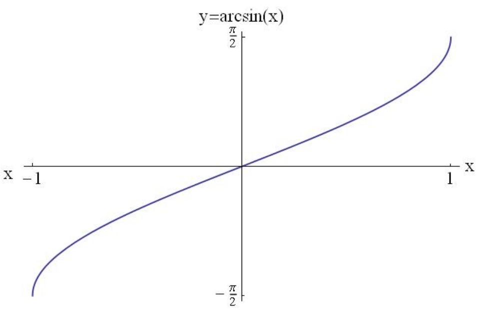
The inverse trig functions are defined on a restricted domain of the original trig functions since the original trig functions are not one-to-one on their entire domain (they fail the horizontal line test) and hence have no inverse.

Notation: $\arcsin x = \sin^{-1} x \neq (\sin x)^{-1} = \dfrac{1}{\sin x}$ and similarly for $\arccos x$ and $\arctan x$.

The Inverse Sine Function

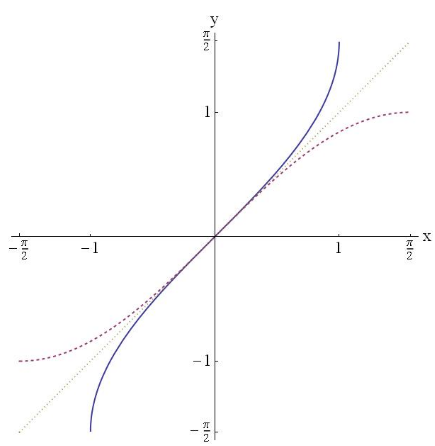




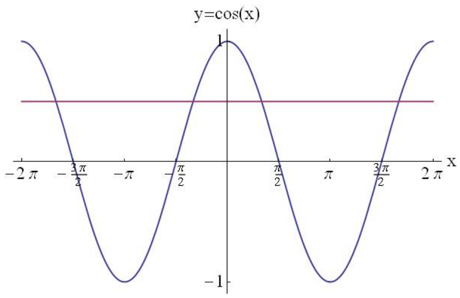


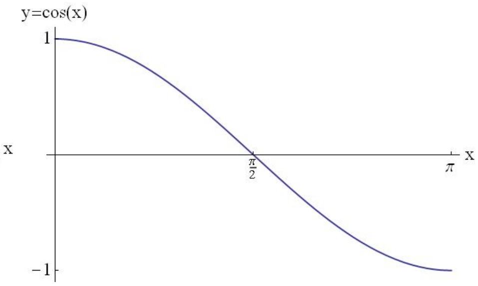
Domain: $x \in [-1, 1]$
Range: $y \in [-\pi/2, \pi/2]$ (so the angle for the inverse sine function is always found in Quadrants I or IV)
Continuity: continuous for all x in domain
Increasing-decreasing behaviour: increasing
Symmetry: odd ($\arcsin(-x) = -\arcsin(x)$))
Boundedness: bounded above and below
Local Extrema: absolute max of $y = \pi/2$, absolute min of $y = -\pi/2$
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: The limits as x approaches $\pm\infty$ do not exist.

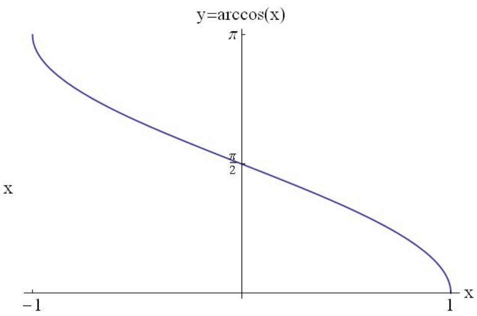
Graphically, the inverse sine function is found from the sine function by reflecting across the line $y = x$.



The Inverse Cosine Function

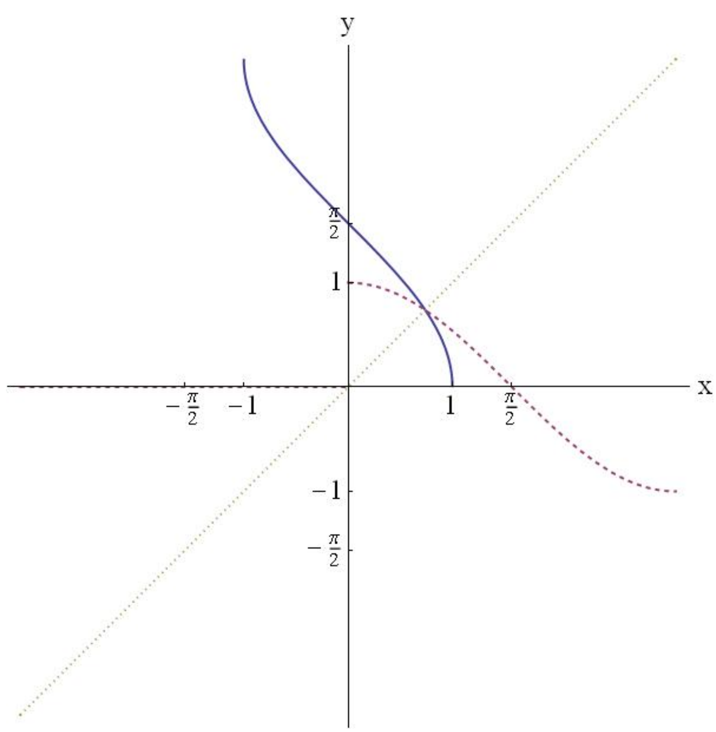




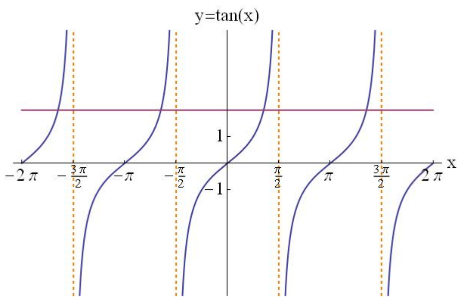


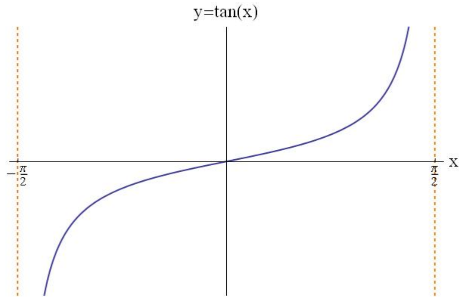
Domain: $x \in [-1, 1]$
Range: $y \in [0, \pi]$ (so, the angle for the inverse cosine function is always found in Quadrants I or II)
Continuity: continuous for all x in domain
Increasing-decreasing behaviour: decreasing
Symmetry: none
Boundedness: bounded above and below
Local Extrema: absolute max of $y = \pi$, absolute min of $y = 0$
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: The limits as x approaches $\pm\infty$ do not exist.

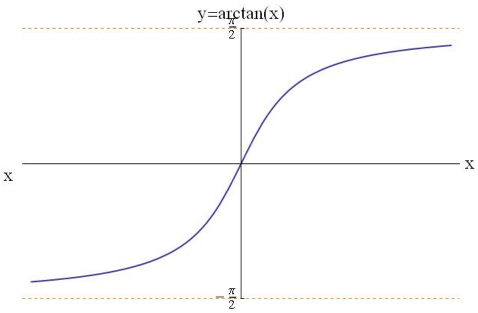
Graphically, the inverse cosine function is found from the cosine function by reflecting across the line $y = x$.



The Inverse Tangent Function

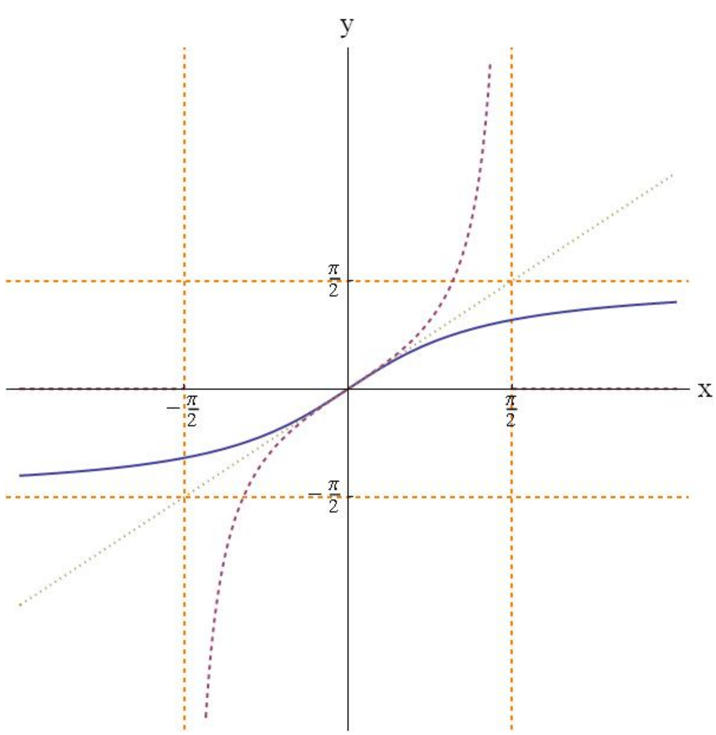






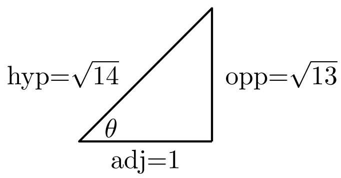
Domain: $x \in \mathbb{R}$
Range: $y \in (-\pi/2, \pi/2)$ (so the angle for the inverse tangent function is always found in Quadrants I or IV)
Continuity: continuous for all x
Increasing-decreasing behaviour: increasing
Symmetry: odd ($\arctan(-x) = -\arctan(x)$))
Boundedness: bounded above and below
Local Extrema: absolute max of $y = \pi/2$, absolute min of $y = -\pi/2$
Horizontal Asymptotes: $y = \pm\pi/2$
Vertical Asymptotes: none
End behaviour: $\lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2}$ $\qquad \lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2}$

Graphically, the inverse tangent function is found from the tangent function by reflecting across the line $y = x$.



Example Find the exact value of $\cos(\arctan\sqrt{13})$ without using a calculator.

To simplify this we need to know the value of $\theta = \arctan\sqrt{13}$.
This means $\tan\theta = \sqrt{13} = \dfrac{\sqrt{13}}{1} = \dfrac{\text{opp}}{\text{adj}}$.
Construct a reference triangle



The length of the hypotenuse was found using the Pythagorean theorem

$$\text{hyp} \;= \sqrt{1^2 + (\sqrt{13})^2} = \sqrt{1+13} = \sqrt{14}.$$

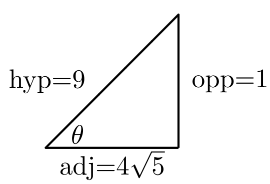
Using the reference triangle, we can determine the value of $\cos\theta$.

$$\cos\big(\arctan(13)\big) = \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{14}}.$$

Notice that we solved this problem without ever figuring out what the angle θ was!

Example Find the exact value of $\tan\theta$ if $\sin\theta = 1/9$, without using a calculator. This is the same as asking for the value of $\tan(\arcsin(1/9))$.

This means $\sin\theta = \dfrac{1}{9} = \dfrac{\text{opp}}{\text{hyp}}$.
Construct a reference triangle



The length of the adjacent side was found using the Pythagorean theorem

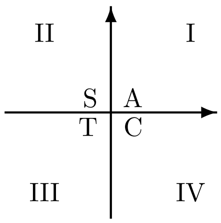
$$\text{adj} \;= \sqrt{9^2 - 1^2} = \sqrt{81-1} = \sqrt{80} = 4\sqrt{5}.$$

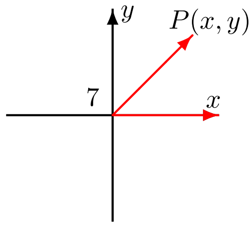
Using the reference triangle, we can determine the value of $\tan\theta$.

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{4\sqrt{5}}.$$

Example Evaluate $\arccos(\cos(7))$ exactly without using a calculator.

To begin, we need to figure out what Quadrant $\cos 7$ is in.
Since $2\pi \leq 7 \leq 5\pi/2$, we are in the first quadrant. The angle 7 actually makes one complete revolution.
The cosine is positive in Quadrant I.





That extra revolution is a problem–remember, we had to restrict the domain of the cosine before we could define the inverse cosine. But, since cosine is periodic, we can use the fact that $\cos 7 = \cos(7 - 2\pi)$, to get back into the correct region.

Therefore, $\arccos(\cos(7)) = \arccos(\cos(7 - 2\pi)) = 7 - 2\pi$, where $0 \leq 7 - 2\pi \leq \pi$ puts our answer in the correct range for an arccosine function.

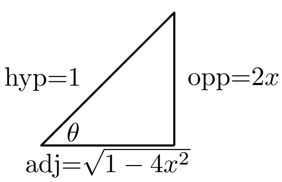
Note: The answer is <u>not</u> 7 (check it on a calculator).

Similarly, you can show $\arccos(\cos(14)) = 14 - 4\pi$ (two revolutions).

Note that you don't have this problem for $\cos(\arccos(7))$, although you have a different problem since $\arccos(7)$ is not a real number!

Example Express $\cos(\arcsin(2x))$ as an algebraic expression involving no trigonometric functions.

To simplify this we need to know the value of $\theta = \arcsin(2x)$.
This means $\sin\theta = 2x = \dfrac{2x}{1} = \dfrac{\text{opp}}{\text{hyp}}$.
Construct a reference triangle



The length of the adjacent side was found using the Pythagorean theorem

$$\text{adj} \;=\; \sqrt{1^2 - (2x)^2} = \sqrt{1 - 4x^2}.$$

$$\cos\big(\arcsin(2x)\big) = \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1 - 4x^2}}{1} = \sqrt{1 - 4x^2}.$$

Notice we did not have to worry about any signs in this problem.