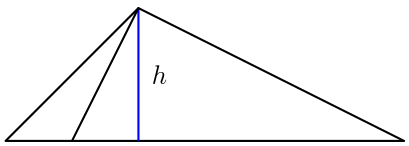
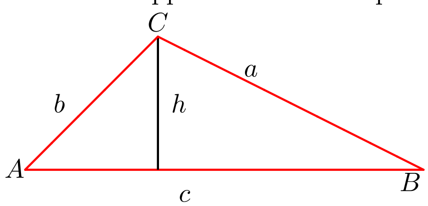
Concepts: Law of Sines, Law of Cosines.

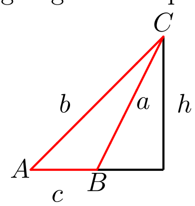
Law of Sines

The law of sines is used to determine all the angles and all the lengths of a general triangle given partial information about the triangle (which is in general not a right-angle triangle).



There are two possibilities for the shape of the triangle created with interior angles A, B, C and sides of length a, b, c. The sides are labelled opposite their corresponding angles. The perpendicular height is labelled h in both cases.





From either of the diagrams above, we have $\sin A = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{h}{b}$.

Also, from the diagram on the left, we have $\sin B = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{h}{a}$.

Also, from the diagram on the right, we have $\sin(\pi - B) = \dfrac{h}{a}$. (remember, B is the *interior angle*!)

We need to show th $\sin B = \sin(\pi - B)$ so we can use the Law of Sines for either situation.

$$\begin{aligned}
\sin(u - v) &= \sin u \cos v - \cos u \sin v \quad \text{start with trig identity} \\
\sin(\pi - B) &= \sin \pi \cos B - \cos \pi \sin B \\
&= (0) \cos B - (-1) \sin B \\
&= \sin B = \frac{h}{a}
\end{aligned}$$

Therefore, for both situations (skinny triangle or fat triangle) we have

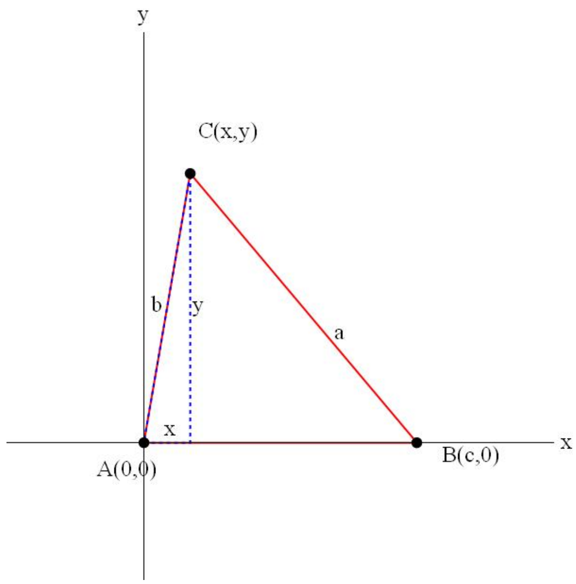
$$h = b \sin A = a \sin B \qquad \Rightarrow \qquad \frac{\sin A}{a} = \frac{\sin B}{b}$$

You could do exactly the same thing where you drop the perpendicular to the other two sides. This leads to the Law of Sines:

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

Law of Cosines

The law of cosines is a generalization of the Pythagorean theorem. It can be derived in a manner similar to how we derived the trig identity $\cos(u - v) = \cos u \cos v + \sin u \sin v$.

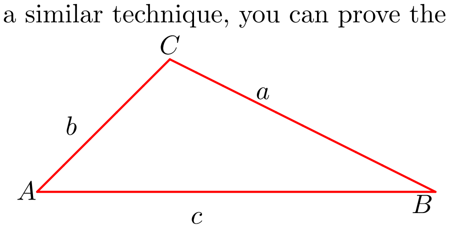


The coordinates of the point C satisfy (remember, A is the interior angle): $\cos A = \dfrac{x}{b}$ and $\sin A = \dfrac{y}{b}$.

Therefore, $x = b\cos A$ and $y = b\sin A$. Using the distance formula, we can write for the distance from C to B:

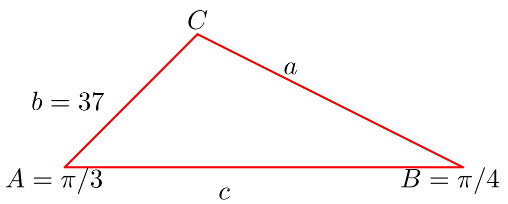
$$\begin{aligned}
a &= \sqrt{(x-c)^2 + (y-0)^2} \\
a^2 &= (x-c)^2 + y^2 \\
a^2 &= (b\cos A - c)^2 + (b\sin A)^2 \\
a^2 &= b^2\cos^2 A + c^2 - 2bc\cos A + b^2\sin^2 A \\
a^2 &= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A \\
a^2 &= b^2(1) + c^2 - 2bc\cos A \\
a^2 &= b^2 + c^2 - 2bc\cos A
\end{aligned}$$

Using a similar technique, you can prove the other law of cosines results.



$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$
$$\boxed{b^2 = a^2 + c^2 - 2ac\cos B}$$
$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

Example Solve the following triangle:



When you know two angles, it is easy to get the third: $C = \pi - \frac{\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$.
Now we can determine a:

$$\begin{aligned}
\frac{\sin A}{a} &= \frac{\sin B}{b} \quad \text{start with a Law of Sines}\\
a &= b\frac{\sin A}{\sin B}\\
&= (37)\frac{\sin \pi/3}{\sin \pi/4}\\
&= (37)\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{\sqrt{2}}\right)}\\
&= \frac{37\sqrt{6}}{2}
\end{aligned}$$

We can determine c:

$$\begin{aligned}
\frac{\sin C}{c} &= \frac{\sin B}{b} \quad \text{start with a Law of Sines}\\
c &= b\frac{\sin C}{\sin B}\\
&= (37)\frac{\sin 5\pi/12}{\sin \pi/4}\\
&= (37)\frac{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)}\\
&= \frac{37(\sqrt{3}+1)}{2}
\end{aligned}$$

Note: The result $\sin 5\pi/12 = \frac{\sqrt{3}+1}{2\sqrt{2}}$ is similar to one of the examples in the lecture notes for Sum and Difference Identities.

$$\frac{5\pi}{12} = \frac{10\pi}{24} = \frac{4\pi}{12} + \frac{6\pi}{24} = \frac{\pi}{6} + \frac{\pi}{4}$$

$$\begin{aligned}
\sin 5\pi/12 &= \sin(\pi/6 + \pi/4) \quad \text{use } \sin(u+v) = \sin u \cos v + \cos u \sin v\\
&= \sin(\pi/6)\cos(\pi/4) + \cos(\pi/6)\sin(\pi/4)\\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

using the special triangles to evaluate the trig functions at $\pi/6$ and $\pi/4$.

The Ambiguous Case

The law of sines allows you to determine completely all the other information in the triangle provided you are given:

- two angles and a side (AAS and ASA).

If you are given two sides and an angle which is not between the two sides (SSA), there might be zero, one, or two triangles determined.

The law of cosines allows you to determine completely all the other information in the triangle provided you are given:

- three sides (SSS),
- two sides of the triangle and the angle between them (SAS).

If you are given two sides and an angle which is not between the two sides (SSA), there might be zero, one, or two triangles determined.

Example Using a calculator to evaluate trig functions, determine if it is possible to draw triangles that have the following characteristics (these are all SSA situations):

a) $A = 30°$, $b = 16$, $a = 6.3$.

b) $A = 46°$, $b = 16$, $a = 18.4$.

c) $A = 36°$, $b = 13$, $a = 10$.

To answer this question, we just need to try to solve each triangle.

(a) Use Law of Sines to get started:

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin B}{b}\\ \sin B &= \frac{b}{a}\sin A\\ &= \frac{16}{6.3}\cdot\sin 30° = 1.26984 > 1 \text{ so no solution } (\sin B \text{ should be less than one})\end{aligned}$$

You cannot construct a triangle with the charachteristics of part (a).

(a) Use Law of Sines to get started:

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin B}{b}\\ \sin B &= \frac{b}{a}\sin A\\ &= \frac{16}{18.4}\cdot\sin 47° = 0.63596\\ B &= \arcsin(0.63596) \sim 39.5°\end{aligned}$$

Then, $C = 180° - 47° - 39.5° = 93.5°$. And finally,

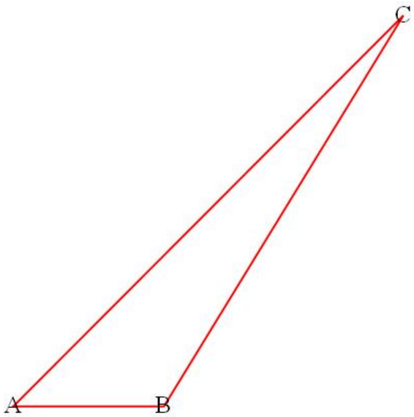
$$c = \sqrt{a^2 + b^2 - 2ab\cos C} = \sqrt{18.4^2 + 16^2 - 2(18.4)(16)\cos 93.5°} = 25.1$$

One triangle has the characteristics of part (b).

(c) Use Law of Sines to get started:

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin B}{b}\\ \sin B &= \frac{b}{a}\sin A\\ &= \frac{13}{10}\cdot\sin 36° = 0.764121\\ B &= \arcsin(0.764121) \sim 49.83°\end{aligned}$$

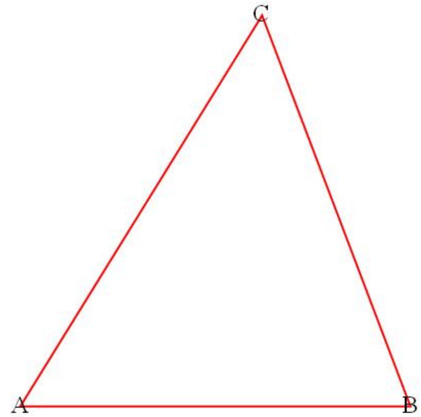
Here's where it gets interesting. Since $b > a$, we can actually construct two triangles that look like this, a skinny one and a fat one (drawn to scale):



$B = 180° - 49.82° = 130.17°$

$C = 180° - 36° - 130.19° = 13.81°$

$$\begin{aligned} c &= \sqrt{a^2 + b^2 - 2ab\cos C} \\ &= \sqrt{10^2 + 13^2 - 2(10)(13)\cos 13.81°} \\ &= 4.06 \end{aligned}$$



$B = 49.82°$

$C = 180° - 36° - 49.82° = 94.18°$

$$\begin{aligned} c &= \sqrt{a^2 + b^2 - 2ab\cos C} \\ &= \sqrt{10^2 + 13^2 - 2(10)(13)\cos 94.18°} \\ &= 16.97 \end{aligned}$$

There are two triangles with the characteristics in (c).