Concepts: Solving Linear Equations, Sketching Straight Lines; Slope, Parallel Lines, Perpendicular Lines, Equations for Straight Lines.

Solving Linear Equations and Inequalities

An *equation* involves an equal sign and indicates that two expressions have the same value.

$x + 42 = 67(4 - x)$ is an equation, and means $x + 42$ has the same value as $67(4 - x)$.

Equivalent equations are equations that have exactly the same solution.

Solving an equation typically involves using the rules of algebra to construct a series of equivalent equations until you determine a numerical solution for an unknown variable. When using algebra, show enough intermediate steps in your solution to get the correct answer. A good rule of thumb is to write enough so that a classmate could read your solution and understand all the steps you used without having you explain it to them.

The Addition Principle: If the same number is added to both sides of an equation, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} x + 42 &= 67(4 - x) \text{ is an equation,} \\ x + 42 + 76 &= 67(4 - x) + 76 \text{ is an equivalent equation.} \end{aligned}$$

The Multiplication Principle: If both sides of an equation are multiplied by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} x + 42 &= 67(4 - x) \text{ is an equation,} \\ 132(x + 42) &= 132 \times 67(4 - x) \text{ is an equivalent equation.} \end{aligned}$$

The Division Principle: If both sides of an equation are divided by the same nonzero number, the results on both sides are equal in value (you have constructed an equivalent equation).

$$\begin{aligned} x + 42 &= 67(4 - x) \text{ is an equation,} \\ \frac{(x + 42)}{69} &= \frac{67(4 - x)}{69} \text{ is an equivalent equation.} \end{aligned}$$

Note that you have to be careful with division and multiplication. Make sure you multiply or divide each entire side of the equation–if you don't, you will be making an algebra error!

Solving an equation of the form $ax + b = cx + d$ (or even slightly more complicated equations) involves constructing a series of equivalent equations that ends with the equivalent equation $x =$ a number. The following steps are required:

1. Clear any parentheses, and simply as much as possible (simplifying is just advice to make things easier).
2. Collect like terms using the Addition Principle if necessary.
3. Isolate the variable term.
4. Use the Division Principle to isolate the variable.
5. Check your answer by substituting back in the original equation to see if your answer is correct.

Literal Equations have many unspecified variables, but you solve them using the same techniques. You can just can't simplify as much since you are working with variables instead of numbers. A nice example of a literal equation used in chemistry is the *Combined Gas Law*, which states that for a gas under two different sets of conditions (labeled by the subscript 1 or 2), it is true that

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}.$$

The process of solution of an *inequality* is the same as for an equation, except that the inequality is reversed if you multiply or divide by a negative number.

When sketching an inequality you use an open circle if the endpoint is not included, and a filled in circle if the endpoint is included. Here's how you can remember this:

For $<$ (one thing)	draw $\circ$	one thing (draw the circle)
For $>$ (one thing)	draw $\circ$	one thing (draw the circle)
For $\geq$ (two things)	draw $\bullet$	two things (draw the circle and then shade it in)
For $\leq$ (two things)	draw $\bullet$	two things (draw the circle and then shade it in)

Interval Notation and Set Notation for Inequalities

$a \leq x \leq b$ is equivalent to $x \epsilon [a, b]$

$a < x < b$ is equivalent to $x \epsilon (a, b)$

$a \leq x < b$ is equivalent to $x \epsilon [a, b)$

$a < x \leq b$ is equivalent to $x \epsilon (a, b]$

Example An ideal gas in state 1 has $P_1 = 3$ Pa, $V_1 = 20$ cm^3, and $T_1 = 40$ K. This gas is then adjusted so the pressure is $P_2 = 4$ Pa and the volume is $V_2 = 50$ cm^3. What is the temperature of the gas in state 2, using the Combined Gas Law?

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \text{ write the equation you will start with}$$

$$\frac{(3\text{Pa})(20\text{cm}^3)}{(40\text{K})} = \frac{(4\text{Pa})(50\text{cm}^3)}{T_2} \text{ substitute in the values}$$

$$T_2 = \frac{(4\text{Pa})(50\text{cm}^3)}{(3\text{Pa})(20\text{cm}^3)}(40\text{K}) \text{ solve for } T_2$$

$$T_2 = \frac{(4\cancel{\text{Pa}})(50\cancel{\text{cm}^3})}{(3\cancel{\text{Pa}})(20\cancel{\text{cm}^3})}(40\text{K}) \text{ solve for } T_2\text{, cancel units}$$

$$T_2 = \frac{(4)(50)}{(3)(20)}(40)\text{K} = \frac{400}{3}\text{K} \sim 133\text{K}$$

I split the canceling of units into it's own step, but it need not be. Show as much detail as you need to get the simplification done correctly.

Example Solve the Combined Gas Law for T_2.

$$\frac{P_1V_1}{T_1} \times T_2 = \frac{P_2V_2}{\cancel{T_2}} \times \cancel{T_2}$$

$$\frac{\cancel{T_1}}{\cancel{P_1V_1}} \times \frac{\cancel{P_1V_1}}{\cancel{T_1}} \times T_2 = \frac{T_1}{P_1V_1} \times P_2V_2$$

$$T_2 = \frac{T_1P_2V_2}{P_1V_1}$$

Example Solve $\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$.

Remember, you might choose a different route to the solution that is entirely correct.
The goal is first to isolate a single term with x in it on one side of the equation.

$$\frac{2}{3}(x+4) = 6 - \frac{1}{4}(3x-2) - 1$$
$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \text{ (I choose to clear parentheses first)}$$
$$\frac{2}{3}x + \frac{8}{3} = 6 - \frac{3}{4}x + \frac{2}{4} - 1 \text{ (simplify on each side of equal side by collecting like terms)}$$
$$\frac{2}{3}x + \frac{8}{3} = \frac{11}{2} - \frac{3}{4}x$$
(use Addition Principle to move all terms with x to left side, all other terms to right side)
$$\frac{2}{3}x + \frac{3}{4}x + \cancel{\frac{8}{3}} - \cancel{\frac{8}{3}} = \frac{11}{2} - \frac{8}{3} - \cancel{\frac{3}{4}x} + \cancel{\frac{3}{4}x}$$
$$\frac{2}{3}x + \frac{3}{4}x = \frac{11}{2} - \frac{8}{3} \text{ (now collect like terms)}$$
$$\frac{17}{12}x = \frac{17}{6} \text{ (now use Multiplication Principle to isolate the } x\text{)}$$
$$\cancel{\frac{12}{17}} \times \cancel{\frac{17}{12}}x = \frac{12}{17} \times \frac{17}{6} \text{ (simplify)}$$
$$x = 2$$

Example Solve $\frac{3x+5}{4} + \frac{7}{12} > -\frac{x}{6}$.

Let's start this one by clearing fractions. So we use the Multiplication Principle with the factor 12 (which is the LCD). Since 12 is positive, we don't change the direction of the inequality.

$$\frac{3x+5}{4} + \frac{7}{12} > -\frac{x}{6}$$
$$12 \times \left(\frac{3x+5}{4} + \frac{7}{12}\right) > 12 \times \left(-\frac{x}{6}\right) \text{ (now distribute the factor of 12)}$$
$$3(3x+5) + 7 > -2x \text{ (distribute the 3)}$$
$$9x + 15 + 7 > -2x \text{ (simplify)}$$
$$9x + 22 > -2x$$
$$9x + 22 - 9x > -2x - 9x \text{ (Use Additive Principle)}$$
$$22 > -11x$$
(Use Multiplication Principle to isolate the x,
since we are multiplying by a negative number change direction of inequality)
$$\frac{1}{-11}22 < \frac{1}{-11}(-11x) \text{ (simplify)}$$
$$-2 < x \text{ (simplify)}$$

Example Solve $5(x-3) \leq 2(x-3)$.

The goal is still to first isolate a single term with the x in it on one side of the equation. If we multiply or divide by a negative number, we must switch the direction of the inequality.

$$\begin{aligned}
5(x-3) &\leq 2(x-3) \\
5x-15 &\leq 2x-6 \\
5x-\cancel{15}+\cancel{15} &\leq 2x-6+15 \\
5x &\leq 2x+9 \\
5x-2x &\leq \cancel{2x}+9-\cancel{2x} \\
3x &\leq 9 \\
\frac{1}{3}\times 3x &\leq \frac{1}{3}\times 9 \\
x &\leq 3
\end{aligned}$$

Example An ideal gas in state 1 has $P_1 = 2$ Pa, $V_1 = 20$ cm^3, and $T_1 = 12$ K. This gas is then adjusted so the temperature is $T_2 = 80$ K and the volume is $V_2 = 10$ cm^3. What is the pressure of the gas in state 2, using the Combined Gas Law?

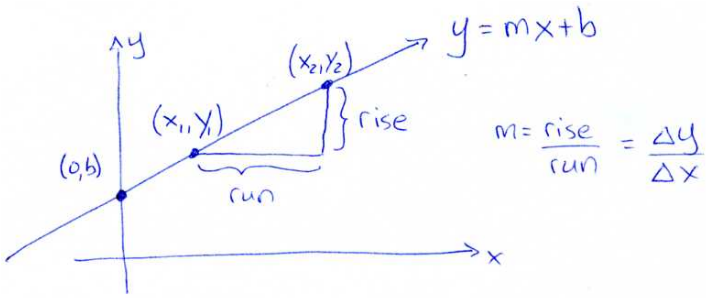
$$\begin{aligned}
\frac{P_1V_1}{T_1} &= \frac{P_2V_2}{T_2} \\
\frac{(2\text{Pa})(20\text{cm}^3)}{(12\text{K})} &= \frac{(P_2)(10\text{cm}^3)}{(80\text{K})} \\
P_2 &= \frac{(2\text{Pa})(20\text{cm}^3)(80\text{K})}{(12\text{K})(10\text{cm}^3)} \\
P_2 &= \frac{3200}{120}\text{Pa} = \frac{80}{3}\text{Pa} \sim 27\text{Pa}
\end{aligned}$$

Example Solve the van der Waals equation (used to model fluid compression in chemistry) $\left(p+\frac{n^2a}{V^2}\right)(v-nb) = nRT$ for p.

$$\begin{aligned}
\left(p+\frac{n^2a}{V^2}\right)(v-nb) &= nRT \\
\left(p+\frac{n^2a}{V^2}\right)\cancel{(v-nb)}\times\frac{1}{\cancel{(v-nb)}} &= nRT\times\frac{1}{(v-nb)} \text{ Division Principle} \\
p+\frac{n^2a}{V^2} &= \frac{nRT}{(v-nb)} \text{ Simplify} \\
p+\cancel{\frac{n^2a}{V^2}}-\cancel{\frac{n^2a}{V^2}} &= \frac{nRT}{(v-nb)}-\frac{n^2a}{V^2} \text{ Addition Principle (adding a negative quantity)} \\
p &= \frac{nRT}{(v-nb)}-\frac{n^2a}{V^2} \text{ Simplify}
\end{aligned}$$

Note: Remember, other paths to the final solution are possible.

Sketching Straight Lines (Linear Relationships)



The *slope* of the line is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}}$.

Horizontal lines have the form $y = b$ and have slope $m = 0$.
Vertical lines have the form $x = a$ and have infinite slope.

Parallel lines have the same slope.
Perpendicular lines have slopes whose product is -1.

The slope of a line represents a rate of change over an interval. This ides is captured in the notation $m = \dfrac{\Delta y}{\Delta x}$, where we read Δy as "the change in y" and Δx as "the change in x". You see this notation in physics and chemistry, where the Δ is used to represent an error in some measured quantity.

The equation of the line $y = mx + b$ represents an infinite set of ordered pairs. We can express this as $(x, y) = (x, mx + b)$ which explicitly shows this as a set of ordered pairs. If we pick a particular value of x, we can evaluate $(x, mx + b)$ to get an ordered pair.

Technique: To find specific ordered pairs, we can

- pick a value of x and use the equation to determine the corresponding value of y, or
- pick a value of y and use the equation to determine the corresponding value of x.

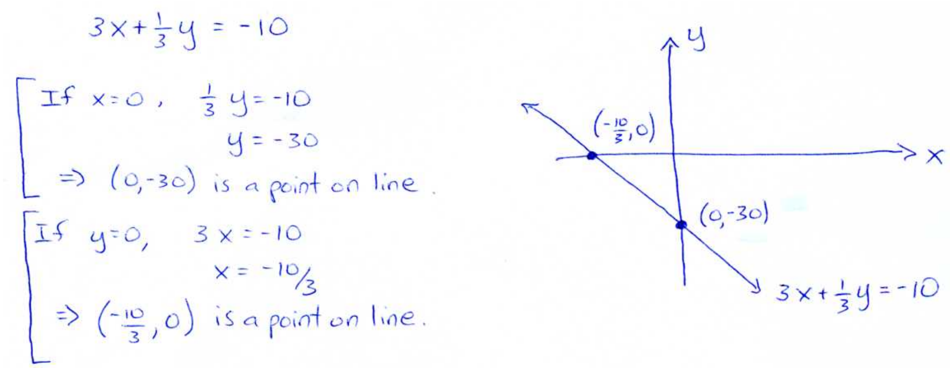
Example Find four ordered pairs that satisfy the equation $3x - 7y = -21$.
If $x = 0$, the the equation becomes $3(0) - 7y = -21 \Rightarrow y = 3$, so an ordered pair on the line is $(x, y) = (0, 3)$.
If $y = 0$, the the equation becomes $3x - 7(0) = -21 \Rightarrow x = -7$, so an ordered pair on the line is $(x, y) = (-7, 0)$.
If $y = -1$, the the equation becomes $3x - 7(-1) = -21 \Rightarrow x = -\frac{28}{3}$, so an ordered pair on the line is $(x, y) = \left(-\frac{28}{3}, -1\right)$.
If $x = 10$, the the equation becomes $3(10) - 7y = -21 \Rightarrow y = \frac{51}{7}$, so an ordered pair on the line is $(x, y) = \left(10, \frac{51}{7}\right)$.

When you are sketching without using graph paper (which we often do), here are some important things to do:

- Label your axes, x axis to the right (in the direction of increasing x) and y axis to the top (in the direction of increasing y).
- Include arrows on the ends of your line if the line continues forever.
- include the equation of the line somewhere on the graph beside the line.
- Explicitly label the points you used to create the line. I prefer not to use ticks on the axes, but you can use ticks if you want–but be neat!
- Any annotations you make on the graph (maybe a triangle that shows the slope between two points on the line) should be large and neatly labeled so it is easy to read.
- Make the entire graph large enough to easily read, and redraw it if necessary.

Technique: Plot two ordered pairs that satisfy the linear equation and draw a straight line through them. A third point that satisfies the linear equation can be used to check that the graph is correct.

Example Graph $3x + \frac{1}{3}y - 2 = -12$.
Solution



One way to check this is to compute the slope from the sketch using the points you used and compare it to the slope from the equation (they should be equal!).

Three different ways of writing the equation of a line

- *slope-intercept form*, where m and y-intercept $(0, b)$ are given: $y = mx + b$.
- *slope-point form*, where m and point on the line (x_1, y_1) are given: $y - y_1 = m(x - x_1)$.
- *point-point form*, where two points on the line (x_1, y_1) and (x_2, y_2) are given: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is sometimes written as $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.

Choose one of the above to work with depending on what information you are given. The first two are the most used in mathematics, and you should strive to be comfortable working with both $y = mx + b$ and $y - y_1 = m(x - x_1)$.

Aside: Here is the derivation of the slope-point equation when you are given m and (x_1, y_1):

$$
\begin{aligned}
y &= mx + b && \text{(begin with slope-intercept equation)} \\
y_1 &= mx_1 + b && \text{(m is known, so substitute in $x = x_1$ and $y = y_1$)} \\
{\color{red} y_1 - mx_1} &= {\color{red} b} && \text{(solve for the unknown b)} \\
y &= mx + {\color{red} y_1 - mx_1} && \text{(substitute this value for b back in the original equation)} \\
y - y_1 &= mx - mx_1 && \text{(simplify)} \\
y - y_1 &= m(x - x_1) && \text{(factor–factoring is covered in Unit 7 in more detail)}
\end{aligned}
$$

The point-point equation follows from the slope-point equation by substituting $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Example Do the points $(2, 1)$, $(-3, -2)$ and $(7, 4)$ lie on the same line? If so, what is the equation of the line?

Solution One way to check if they all lie on the same line is to check that the slope between all pairs of points is the same. Draw three points that aren't on the same line and see that this must the case.
Slope between $(2, 1)$ and $(-3, -2)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{-2-1}{-3-2} = \dfrac{-3}{-5} = \dfrac{3}{5}$.
Slope between $(2, 1)$ and $(7, 4)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{4-1}{7-2} = \dfrac{3}{5}$.
Slope between $(-3, -2)$ and $(7, 4)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{4-(-2)}{7-(-3)} = \dfrac{6}{10} = \dfrac{3}{5}$.
Since the slopes between each pair of points is the same, the points all lie on the same line.
We can get the equation of the line using any of the formulas for equation of a line, so let's do that here so you can see how each gives the same final equation.
slope-intercept form:

$$\begin{aligned}
y &= mx + b \\
y &= \frac{3}{5}x + b \quad \text{substitute in the slope which we worked out above} \\
(1) &= \frac{3}{5} \cdot (2) + b \quad \text{substitute in one of the points, here I've chosen } (x, y) = (2, 1) \\
1 - \frac{6}{5} &= b \quad \text{solve for } b \\
-\frac{1}{5} &= b \\
y &= \frac{3}{5}x - \frac{1}{5} \quad \text{final equation for the line}
\end{aligned}$$

slope-point form: Use $(x_1, y_1) = (2, 1)$ and $m = \frac{3}{5}$:

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 1 &= \frac{3}{5}(x - 2) \quad \text{substitute in the slope and point} \\
y &= \frac{3}{5}x - \frac{6}{5} + 1 \quad \text{simplify to slope-intercept form to compare} \\
y &= \frac{3}{5}x - \frac{1}{5}
\end{aligned}$$

point-point form: Use $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (7, 4)$:

$$\begin{aligned}
\frac{y - y_1}{y_2 - y_1} &= \frac{x - x_1}{x_2 - x_1} \\
\frac{y - 1}{4 - 1} &= \frac{x - 2}{7 - 2} \quad \text{substitute in the points} \\
\frac{y - 1}{3} &= \frac{x - 2}{5} \quad \text{simplify to slope-intercept form to compare} \\
15 \times \left(\frac{y - 1}{3}\right) &= 15 \times \left(\frac{x - 2}{5}\right) \quad \text{(clear fractions)} \\
5\,(y - 1) &= 3\,(x - 2) \\
5y - 5 &= 3x - 6 \\
5y &= 3x - 6 + 5 \\
5y &= 3x - 1 \\
y &= \frac{3}{5}x - \frac{1}{5}
\end{aligned}$$

Example The amount of debt outstanding on home equity loans in the USA during the period from 1993 to 2008 can be approximated by the equation $y = mx + b$, where x is the number of years since 1993 and y is the debt measured in billions of dollars. Find the equation if two ordered pairs that satisfy it are $(1, 280)$ and $(6, 500)$.

Solution We ultimately want a slope-intercept equation of a line, but we are given two points that lie on the line. Let's start with the point-point equation of a line, and use algebra to reduce it to a slope-intercept form. Choose $(x_1, y_1) = (1, 280)$ and $(x_2, y_2) = (6, 500)$.

$$\begin{aligned}
\frac{y - y_1}{y_2 - y_1} &= \frac{x - x_1}{x_2 - x_1} \\
\frac{y - 280}{500 - 280} &= \frac{x - 1}{6 - 1} \\
\frac{y - 280}{220} &= \frac{x - 1}{5} \\
220 \times \left(\frac{y - 280}{220}\right) &= 220 \times \left(\frac{x - 1}{5}\right) \quad \text{(clear fractions)} \\
y - 280 &= 44\,(x - 1) \\
y - 280 &= 44x - 44 \\
y &= 44x - 44 + 280 \\
y &= 44x + 236
\end{aligned}$$

Example A student organization sells t-shirts. When they charge \$15 per shirt, they sell 100 shirts. When they charge \$12 per shirt, they sell 175 shirts. Find a linear relation between the price of the shirts x and the number of shirts that are sold y.

Solution Let the relation be $y = mx + b$. Our job is to figure out m and b.
Two points on the line are $(15, 100)$ and $(12, 175)$.
Slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{100 - 175}{15 - 12} = \dfrac{-75}{3} = -25$.
Get b, the y-intercept:

$$\begin{aligned}
y &= mx + b \\
y &= -25x + b \\
100 &= -25(15) + b \text{ sub in a point on the line} \\
100 &= -375 + b \text{ solve for } b \\
475 &= b
\end{aligned}$$

Relation is $y = -25x + 475$, or (number of shirts sold) $= -25$(price of shirt) $+ 475$.

Example Find the equation of the line perpendicular to the line $y = 3x$ that passes through the point $(-2, 1)$. Sketch the situation.

Solution The new line should have slope $-\frac{1}{3}$ (perpendicular, so product of slopes should be -1).
Thus, the new line should look like $y = -\frac{1}{3}x + b$.
Now, use the point given to get the value of b.

$$\begin{aligned}
y &= -\frac{1}{3}x + b \\
1 &= -\frac{1}{3}(-2) + b \\
1 &= \frac{2}{3} + b \\
1 - \frac{2}{3} &= b \\
\frac{1}{3} &= b
\end{aligned}$$

$y = -\frac{1}{3}x + \frac{1}{3}$ is the equation of the line we seek.

sketch $y=3x$

If $x=0$, $y=3(0)$
$y=0$
so $(0,0)$ is a point on the line.

choosing $y=0$ results in the same point, $(0,0)$, so use slope to get second point.

sketch $y=-\frac{1}{3}x+\frac{1}{3}$

If $x=0$, $y=-\frac{1}{3}(0)+\frac{1}{3}$
$y=\frac{1}{3}$
so $(0,\frac{1}{3})$ is a point on the line.

If $y=0$, $0=-\frac{1}{3}x+\frac{1}{3}$
$+\frac{1}{3}x=\frac{1}{3}$
$x=1$
so $(1,0)$ is a point on the line.

when sketching two lines that are supposed to be perpendicular, you should try to make the scale along x and y axes the same (or use graph paper).