Concepts: definition of polynomial functions, linear functions (three representations), transformation of $y = x$ to get $y = mx + b$, quadratic functions (axis of symmetry, vertex, x-intercepts), transformations of $y = x^2$ to get vertex form $y = a(x - h)^2 + k$, sketches.

Polynomial Functions

Let n be a nonnegative integer and let a_0, a_1, $\ldots$,a_{n-1}, a_n be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function of degree n. The leading coefficient is a_n.

Linear Functions

Linear functions are discussed in Sections P.3 and P.4 in the prerequisites.

Three different ways of writing the equation of a line:

- slope-intercept form, where m and y-intercept $(0, b)$ are given: $y = mx + b$.
- slope-point form, where m and point on the line (x_1, y_1) are given: $y - y_1 = m(x - x_1)$. (CALCULUS)
- point-point form, where two points on the line (x_1, y_1) and (x_2, y_2) are given:
  $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is sometimes written as $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.

Choose one of the above to work with depending on what information you are given. I suggest memorizing all three formulas, although you could always work from $y = mx + b$ if you like.

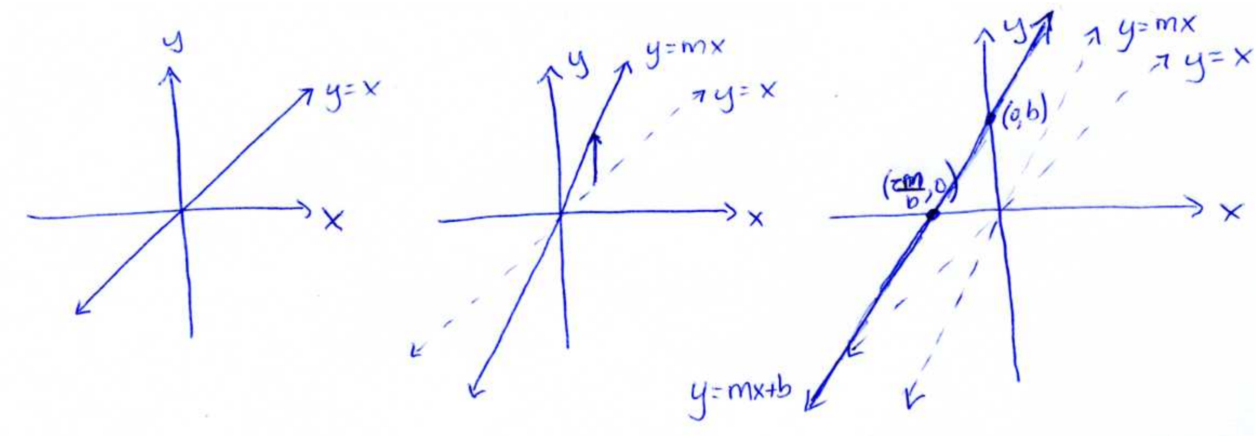
Basic Function: Identity Function $f(x) = x$

This is a linear function with slope $m = 1$ and y-intercept 0.

Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}$
Continuity: continuous for all x
Increasing-decreasing behaviour: increasing for all x
Symmetry: odd
Boundedness: not bounded
Local Extrema: none
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: $\lim\limits_{x \to \infty} x = \infty$ and $\lim\limits_{x \to -\infty} x = -\infty$

Transformations of Identity Function

Basic function: $y = x$
Vertical stretch of m units ($m > 1$): $y = mx$
Shift up b units ($b > 0$): $y = mx + b$.



Example Write an equation for the linear function f such that $f(-1) = 2$ and $f(3) = -2$.

The two points the function must pass through are $(x, f(x)) = (-1, 2) = (x_0, y_0)$, and $(x, f(x)) = (3, -2) = (x_1, y_1)$.
The linear function is given by $y = f(x) = ax + b$, or in point slope form by $y - y_1 = m(x - x_1)$.
The slope of the line is $m = a = \text{rise/run} = (2 - (-2))/(-1 - 3) = 4/(-4) = -1$.
Use the point-slope form of the line $y - (-2) = (-1)(x - 3) \longrightarrow y = -x + 3 - 2 = -x + 1$.
The linear function is given by $f(x) = -x + 1$.

Alternate solution:

The two points the function must pass through are $(x, f(x)) = (-1, 2) = (x_0, y_0)$, and $(x, f(x)) = (3, -2) = (x_1, y_1)$.
The linear function is given by $y = f(x) = ax + b$, or in point-point form by $\dfrac{y - y_1}{x - x_1} = \dfrac{y_0 - y_1}{x_0 - x_1}$.

$$\begin{aligned}
\frac{y - y_1}{x - x_1} &= \frac{y_0 - y_1}{x_0 - x_1} \\
\frac{y - (-2)}{x - 3} &= \frac{2 - (-2)}{-1 - 3} \\
\frac{y + 2}{x - 3} &= -1 \\
y + 2 &= -(x - 3) \\
y &= -x + 3 - 2 = -x + 1
\end{aligned}$$

The linear function is given by $f(x) = -x + 1$.

Quadratic Functions

A quadratic function is a polynomial of degree 2, $y = ax^2 + bx + c$.

It takes three points to determine the three constants a, b, and c in a quadratic function.

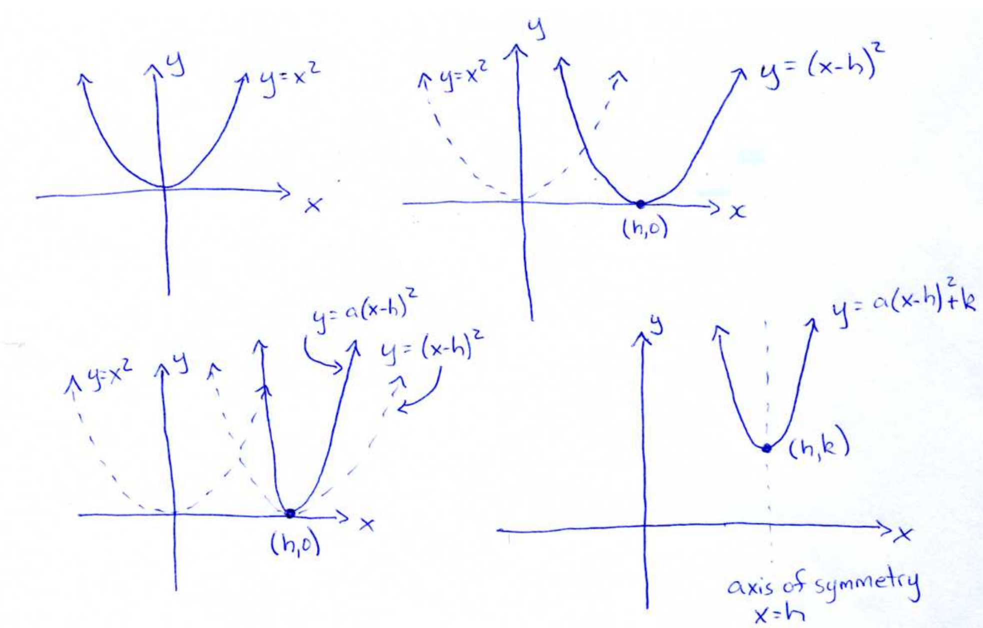
Quadratic functions are described by their axis of symmetry and vertex.

Basic Function: Squaring Function $f(x) = x^2$

Domain: $x \in \mathbb{R}$
Range: $y \in [0, \infty)$
Continuity: continuous for all x
Increasing-decreasing behaviour: decreasing for $x < 0$, increasing for $x > 0$
Symmetry: even (axis of symmetry is the line $x = 0$)
Boundedness: bounded below
Local Extrema: minimum at $(0, 0)$ (the vertex)
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to-\infty} x^2 = \infty$

Transformations of Squaring Function

Basic function: $y = x^2$
Shift right by h units ($h > 0$): $y = (x - h)^2$
Stretch vertically by a units ($a > 1$): $y = a(x - h)^2$
Shift up k units ($k > 0$): $y = a(x - h)^2 + k$.



The Vertex Form

The form $f(x) = a(x - h)^2 + k$ is called the vertex form for a quadratic function.

The vertex of the parabola is (h, k).
The axis of symmetry is $x = h$.
If $a > 0$, the parabola opens up, if $a < 0$ the parabola opens down.

x-intercepts

The x-intercepts of the quadratic function are found from the solution to the quadratic equation: $f(x) = ax^2 + bx + c = 0$, which can be determined using completing the square (c.f. page 46) as

$$x\text{-intercepts} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example Find the axis of symmetry and vertex of the quadratic function $f(x) = 3x^2 + 5x - 4$ by writing in the vertex form. Then find the x-intercepts and sketch the function.

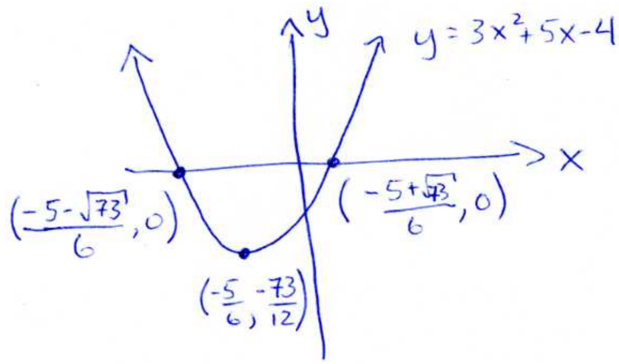
To solve this problem we need to write the quadratic function in vertex form. We can do this by completing the square.

$$\begin{aligned}
3x^2 + 5x &= 3\left(x^2 + \frac{5}{3}x\right) \\
&= 3\left(x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2\right) - 3\left(\frac{5}{6}\right)^2 \\
&= 3\left(x + \frac{5}{6}\right)^2 - \frac{75}{36} \\
f(x) &= 3x^2 + 5x - 4 \\
&= 3\left(x + \frac{5}{6}\right)^2 - \frac{75}{36} - 4 \\
&= 3\left(x + \frac{5}{6}\right)^2 - \frac{75}{36} - \frac{144}{36} \\
&= 3\left(x + \frac{5}{6}\right)^2 - \frac{219}{36} \\
&= 3\left(x - \left(-\frac{5}{6}\right)\right)^2 - \frac{73}{12}
\end{aligned}$$

From this form we can identify the vertex as $(-5/6, -73/12)$, and the axis of symmetry as $x = -5/6$. The quadratic function opens up since the leading coefficient is $3 > 0$.

The x-intercepts are found using the quadratic formula to solve $3x^2 + 5x - 4 = 0$, or since we have the vertex form we can get them directly:

$$\begin{aligned}
3\left(x + \frac{5}{6}\right)^2 - \frac{73}{12} &= 0 \\
\left(x + \frac{5}{6}\right)^2 &= \frac{73}{36} \\
x + \frac{5}{6} &= \pm\sqrt{\frac{73}{36}} \\
x &= -\frac{5}{6} \pm \frac{\sqrt{73}}{6} \\
&= \frac{-5 \pm \sqrt{73}}{6}
\end{aligned}$$



The Average Rate of Change

The average rate of change of the linear function $f(x) = mx + b$ over the interval $(x, x + h)$ is the slope m.

$$\text{average rate of change } = \frac{f(x+h) - f(x)}{h} = \frac{(m(x+h)+b) - (mx+b)}{h} = \frac{mx + mh + b - mx - b}{h} = \frac{mh}{h} = m.$$

The average rate of change here does not depend on the interval $(x, x + h)$.

The average rate of change of the quadratic function $f(x) = ax^2 + bx + c$ on the interval $(x, x + h)$ is:

$$\begin{aligned}
\text{average rate of change} &= \frac{f(x+h) - f(x)}{h} = \frac{(a(x+h)^2 + b(x+h) + c) - (ax^2 + bx + c)}{h} \\
&= \frac{\cancel{ax^2} + ah^2 + 2ahx + \cancel{bx} + bh + \cancel{c} - \cancel{ax^2} - \cancel{bx} - \cancel{c}}{h} \\
&= \frac{ah^2 + 2ahx + bh}{h} = \frac{\cancel{h}(ah + 2ax + b)}{\cancel{h}} = ah + 2ax + b
\end{aligned}$$

Notice that the average rate of change here depends on the interval $(x, x + h)$.