Concepts: Logarithmic Functions, Laws of Exponents, Laws of Logarithms, The Natural Logarithm, Transformation of Logarithm function.

As we saw earlier, if $b > 0$ and $b \neq 1$, the exponential function $y = b^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test.

Therefore, it has an inverse, f^{-1} (Review Section 1.5) which is called the *logarithmic function with base b*. Using our definition of inverse functions, we have

$$\log_b(x) = y \longleftrightarrow b^y = x.$$

So, if $x > 0$, then $\log_b x$ is the exponent to which the base b must be raised to give x.

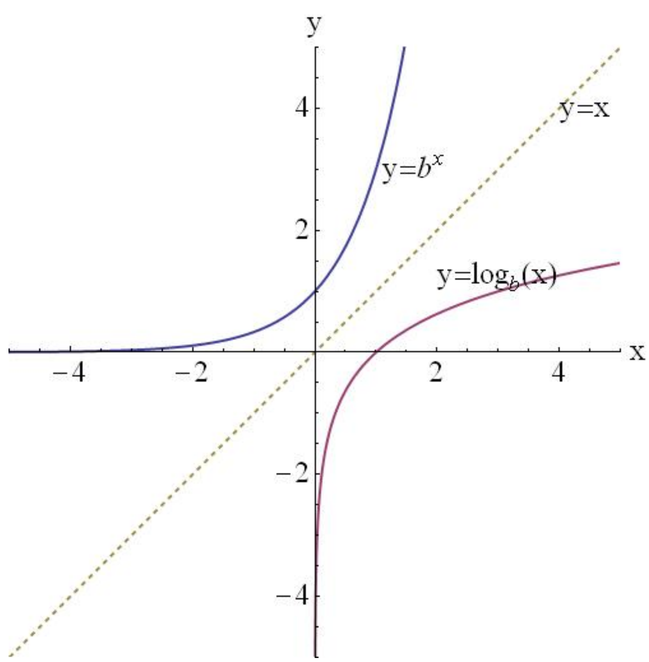
For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The inverse function cancellation equations can be written for the logarithmic and exponential functions as:

$$\log_b(b^x) = x \text{ for every } x \in (-\infty, \infty)$$
$$b^{\log_b(x)} = x \text{ for every } x \in (0, \infty)$$

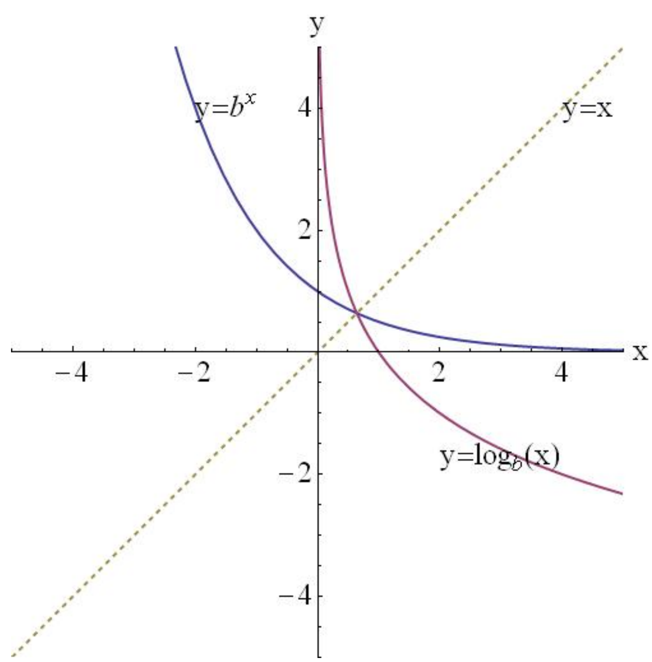
The domains and ranges are apparent if we look at the graphs of b^x and $\log_b x$:

$b > 1$



$\lim_{x\to\infty} \log_b x = \infty$
$\lim_{x\to 0^+} \log_b x = -\infty$
$\lim_{x\to\infty} b^x = \infty$
$\lim_{x\to -\infty} b^x = 0$

$b < 1$



$\lim_{x\to\infty} \log_b x = -\infty$
$\lim_{x\to 0^+} \log_b x = \infty$
$\lim_{x\to\infty} b^x = 0$
$\lim_{x\to -\infty} b^x = \infty$

Laws of Exponents If x and y are real numbers, and $b > 0$ is real, then

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$

Laws of Logarithms If x and y are positive numbers, and $b > 0, b \neq 1$ is real, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ where r is any real number

The Common Logarithm: Base 10

Logarithms with base 10 are very common, since we are used to working with base 10 in almost all the math we do. Hence they are called the common logarithms. We write

$$\log_{10} x = \log x$$

(drop the base 10) since it is understood that the base is 10. We have

$$y = \log x \longleftrightarrow 10^y = x.$$

Inverse function cancelation equations:

$$\log 10^x = x, x \in (-\infty, \infty)$$
$$10^{\log x} = x, x > 0$$

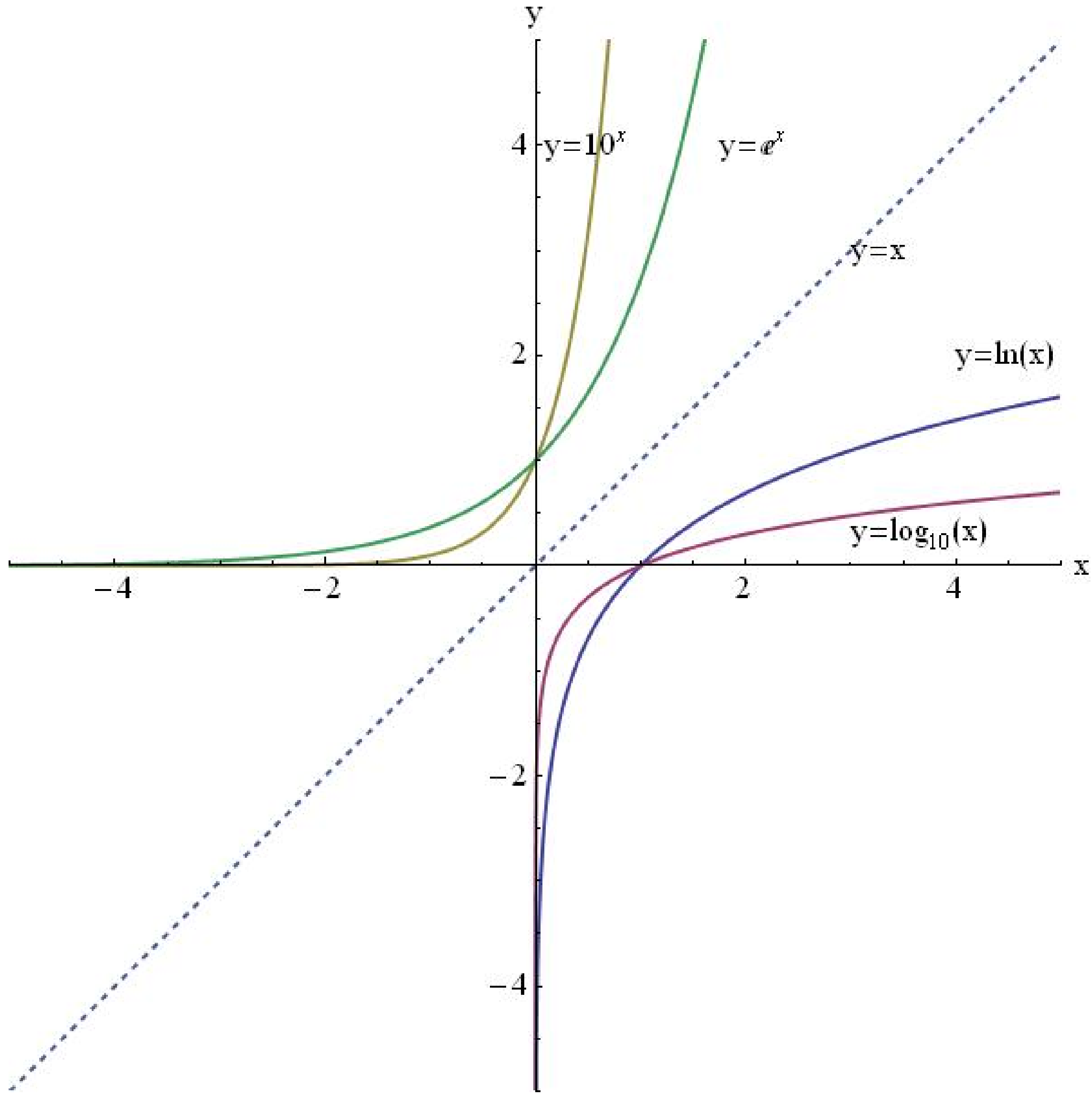
The Natural Logarithm: Base e

If the base that is used is e, from the natural exponential function, we have a natural logarithm.

We write $\log_e x = \ln x$ if $y = \ln x \longleftrightarrow e^y = x$. The cancelation equations are:

$$\begin{aligned} \ln(e^x) &= x \qquad x \in (-\infty, \infty) \\ e^{\ln x} &= x \qquad x > 0 \end{aligned}$$

Here is a sketch of the common and natural logarithms, and their inverses the exponential functions:



From the sketch, you can work out properties of these functions, such as the following for $y = f(x) = \ln x$:

Domain: $x \in (0, \infty)$
Range: $y \in \mathbb{R}$
Continuous on $(0, \infty)$
Increasing on $(0, \infty)$
No symmetry
Not bounded
No local extrema
No horizontal asymptotes
Vertical asymptote: $\lim_{x \to 0^+} \ln x = -\infty$
End Behaviour: $\lim_{x \to \infty} \ln x = \infty$

Example Write $\log_7 x$ in terms of common and natural logarithms.

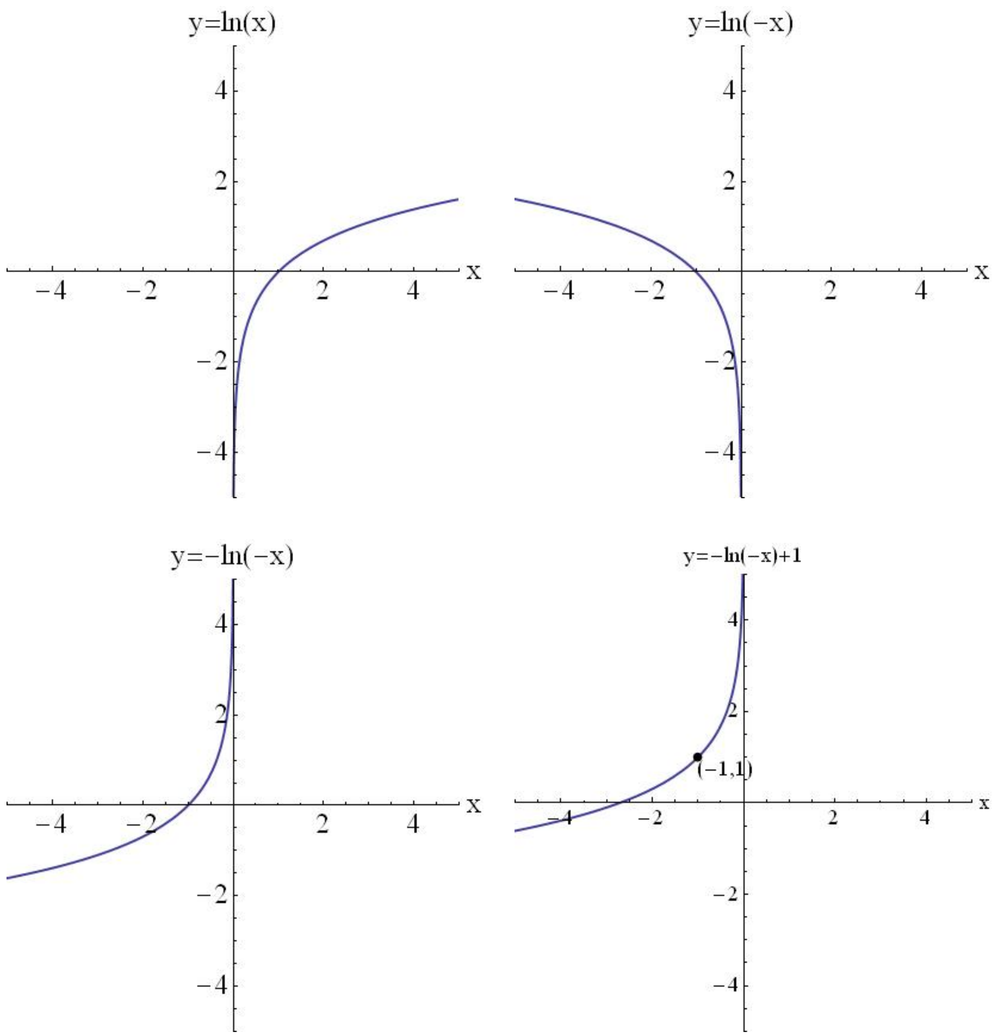
Convert to common logarithms:
Let $y = \log_7 x \longrightarrow 7^y = x$.

$$\begin{aligned} 7^y &= x \\ \log(7^y) &= \log x \\ y\log(7) &= \log x \\ y &= \frac{\log x}{\log 7} \\ \log_7 x &= \frac{\log x}{\log 7} \end{aligned}$$

Convert to natural logarithms:
Let $y = \log_7 x \longrightarrow 7^y = x$.

$$\begin{aligned} 7^y &= x \\ \ln(7^y) &= \ln x \\ y\ln(7) &= \ln x \\ y &= \frac{\ln x}{\ln 7} \\ \log_7 x &= \frac{\ln x}{\ln 7} \end{aligned}$$

Example Describe how to transform the graph of $y = \ln x$ into the graph of $f(x) = -\ln(-x) + 1$.



Example Solve $e^{5-3x} = 10$ for x.

Take the natural logarithm of both sides:

$$\begin{aligned}
\ln(e^{5-3x}) &= \ln 10 \\
5 - 3x &= \ln 10 \\
-3x &= \ln 10 - 5 \\
x &= \frac{\ln 10 - 5}{-3} \\
x &= \frac{5 - \ln 10}{3}
\end{aligned}$$

Example Given $f(x) = 3e^{x+2}$, find $f^{-1}(x)$.

$$\begin{aligned}
\text{Let: } y &= 3e^{x+2} \\
\text{Interchange } x \text{ and } y\text{: } x &= 3e^{y+2} \\
\text{Solve for } y\text{: } \ln\left(\frac{x}{3}\right) &= \ln(e^{y+2}) \\
\ln\left(\frac{x}{3}\right) &= y + 2 \\
y &= f^{-1}(x) = \ln\left(\frac{x}{3}\right) - 2
\end{aligned}$$

Verify the cancelation equations:

$$\begin{aligned}
f(f^{-1}(x)) &= f\left(\ln\left(\frac{x}{3}\right) - 2\right) \\
&= 3\exp\left(\ln\left(\frac{x}{3}\right) - 2 + 2\right) \\
&= 3\exp\left(\ln\left(\frac{x}{3}\right)\right) \\
&= 3\left(\frac{x}{3}\right) \\
&= x
\end{aligned}$$

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}(3e^{x+2}) \\
&= \ln\left(\frac{3e^{x+2}}{3}\right) - 2 \\
&= \ln\left(e^{x+2}\right) - 2 \\
&= x + 2 - 2 \\
&= x
\end{aligned}$$