Concepts: relations, explicit functions, implicit functions, parametric functions, one-to-one functions, finding the inverse of a function (both algebraically and graphically), inverse function cancelation equations.

Explicit, Implicit, and Parametric Relations

A *relation* is another term for a set of ordered pairs (x, y).

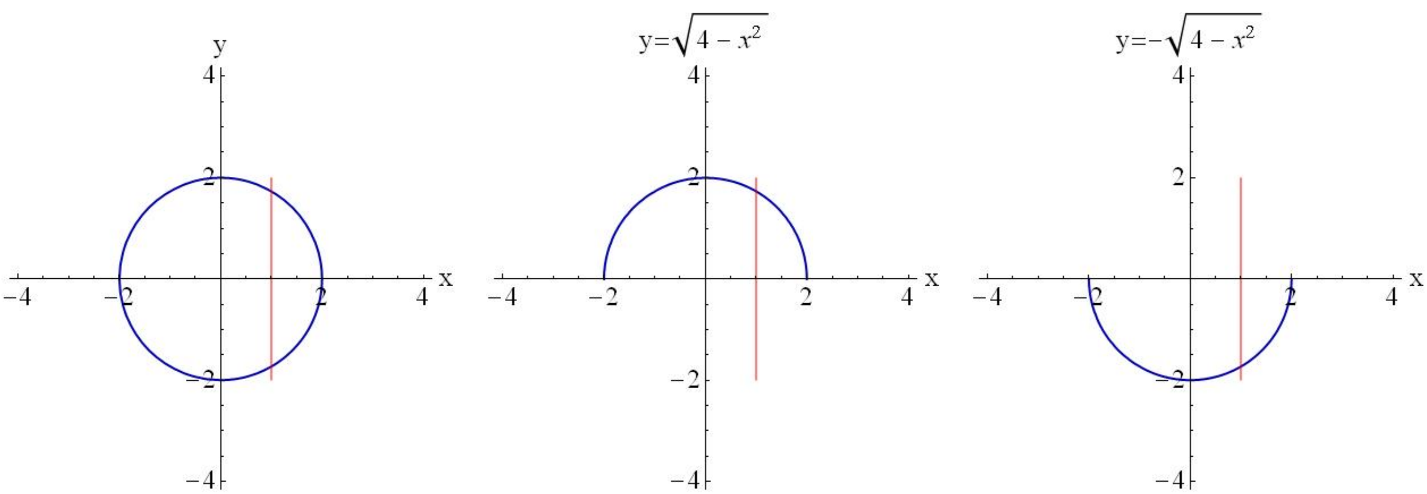
The set of ordered pairs were used to create a graph of a function. We can also create graphs of relations which aren't functions.

Explicitly defined functions are what we have been working with so far, $y = f(x)$. The ordered pair (x, y) is obtained easily as $(x, f(x))$.

A rule which may not be a function can still define a relation. For example, consider the rule

$$x^2 + y^2 = 4.$$

This defines a set of ordered pairs (x, y) which we can sketch.



Implicit Function Explicit Function Explicit Function

This relation does not, however, define a function, since the sketch will fail the vertical line test. However, we can always break a relation like this up into many parts which will pass the vertical line test, and these many parts will then be functions.

$$y = \sqrt{4 - x^2}, \qquad y = -\sqrt{4 - x^2}.$$

In this way, a relation of this type defines a group of functions implicitly, and we say that these are *implicit functions*. Note that although it is theoretically possible to find all the functions that are represented, it may be impossible to actually find them!

A third way of defining relations is *parametrically*. In this case the ordered pair (x, y) is defined in terms of a parameter t, such as

$$x = t + 1, \qquad y = t^2 + 2t.$$

For different values of the parameter t, we get different ordered pairs (x, y).

Inverse Relations and Functions

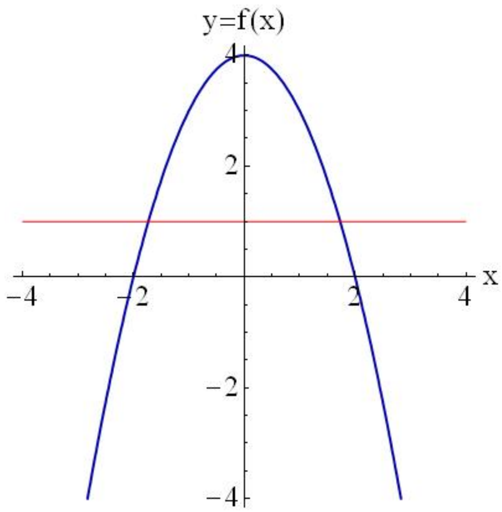
A relation (a, b) has an *inverse relation* defined by the set of ordered pairs (b, a).

Thus, if we had a relation $(x, x^3 - 4)$, $0 \leq x \leq 1$, the inverse relation would be $(x^3 - 4, x)$, $0 \leq x \leq 1$.

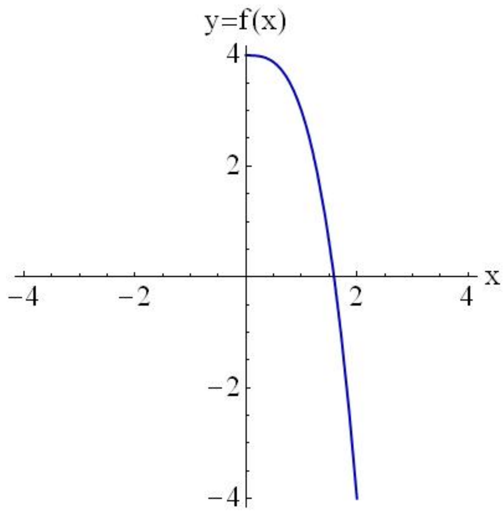
We can tell from a graph whether the inverse relation will be a function by using the *horizontal line test*, which states that the inverse of a relation is a function if and only if each horizontal line intersects the graph of the original relation in at most one point.

Not all functions have inverses. Recall that a function is defined as a rule f that assigns to each element x in a set A exactly one element, called $f(x)$ in a set B. This requirement that a function have exactly one element imposes a condition on whether or not a function f has an inverse function.

You can check if a function will have an inverse using the horizontal line test.



f has no inverse



f has an inverse

Definition A function f is called a *one-to-one* function if it never takes on the same value twice, that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

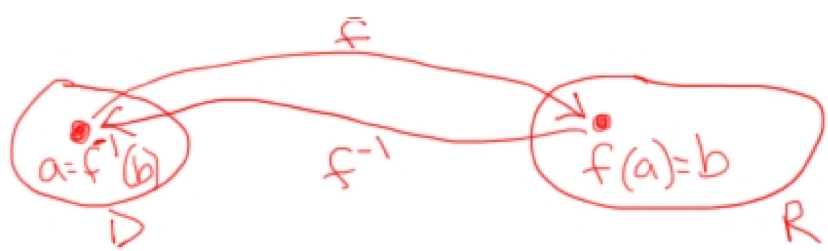
You can check whether or not a function is one-to-one from its graph by using the horizontal line test.

Definition Let f be a one-to-one function with domain D and range R. Then its *inverse function* f^{-1} has domain R and range D and is defined by:

$$f^{-1}(b) = a \text{ if and only if } f(a) = b$$

for any b in R.

Arrow Diagram Note that the domain of f^{-1} is the range of f; and that the range of f^{-1} is the domain of f.



DANGER!! The notation for inverse may be a bit confusing.

$f^{-1}(y)$ DOES NOT MEAN $\dfrac{1}{f(y)}$, $\qquad \dfrac{1}{f(y)} = [f(y)]^{-1}$

Cancelation Equations, or Inverse Composition Rules

$f^{-1}(f(x)) = x$ for every x in D

$f(f^{-1}(x)) = x$ for every x in R

Example of Cancelation equations $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$:

$f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x$

$f(f^{-1}(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$

How to find the inverse function of a one-to-one function Algebraically

Step 1 Write $y = f(x)$. Check for restrictions on the domain to ensure the function is one-to-one.

Step 2 Interchange x and y in the formula $y = f(x)$.

Step 3 Solve this equation for y (if possible).

Example of computing an inverse function Find the inverse function of $f(x) = \dfrac{x-1}{x} + 2$, $x \in (-\infty, 0)$.

STEP 1: $y = \dfrac{x-1}{x} + 2$

STEP 2: $x = \dfrac{y-1}{y} + 2$

STEP 3:

$$\begin{aligned} x &= \frac{y-1}{y} + 2 \\ x &= 1 - \frac{1}{y} + 2 \\ x &= -\frac{1}{y} + 3 \\ \frac{1}{y} &= 3 - x \\ y = f^{-1}(x) &= \frac{1}{3-x} \end{aligned}$$

Domain of f is $x \in (-\infty, 0)$, range of f is $y \in (3, \infty)$.
(we will see how to get the range from a hand sketch in Section 1.6, but you can figure out the range using $f(x) = 3-1/x$, end behaviour, and what happens as you approach $x = 0$).
Domain of f^{-1} is $x \in (3, \infty)$, range of f^{-1} is $y \in (-\infty, 0)$

Inverse from a Graph By the definition of an inverse relation, we get the inverse of the relation (a, b) by plotting the points (b, a). But we get the point (b, a) from reflecting the point (a, b) about the line $y = x$.

Technique The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example of inverse from Graph Sketch the graph of $f(x) = \sqrt{x}$ and its inverse.