Concepts: Vertical Asymptotes, Transforming Reciprocal Function, Sketching Rational Functions,

A *rational function* is a function which is a ratio of two polynomials g and h: $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \neq 0$.

Since polynomials are continuous functions, the domain of a rational function is all $x \in \mathbb{R}$ except possibly at values of x for which the denominator is zero. Where the denominator goes to zero it is possible to have a vertical asymptote of the function.

The function f has a *vertical asymptote* at $x = a$ if:

$$\lim_{x \to a^-} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

New notation: Limits from the left and right:
$\lim\limits_{x \to a^-} f(x)$ means the limit as x approaches a from the left (negative side of the x axis from a),
$\lim\limits_{x \to a^+} f(x)$ means the limit as x approaches a from the right (positive side of the x axis from a).

Sketching a Ratio of Linear Functions by Transforming the Reciprocal Function

If the rational function f is a ratio of <u>linear functions</u>, $f(x)$ can be sketched by finding how f is transformed from the reciprocal function.

Although this is an interesting use of the ideas of graphical transformation, it is usually easier to just use the techniques for sketching a rational function in general. I include this example solely as a example of graphical transformations in action.

Example Sketch the graph of $f(x) = \dfrac{2x-5}{6x-12}$ by transforming the reciprocal function.

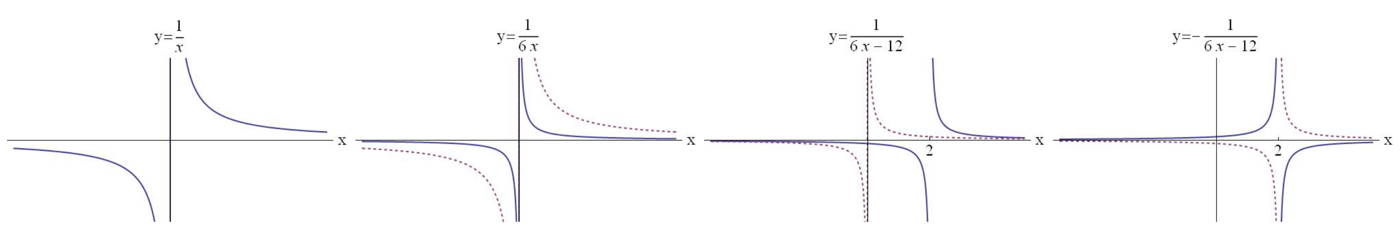
First, we must use long division to write $f(x)$ with a degree 1 polynomial in the denominator and a constant in the numerator

$$\begin{array}{r} 1/3 \\ 6x-12 \,\big)\overline{\,2x-5\,} \\ \underline{2x-4} \\ -1 \end{array}$$

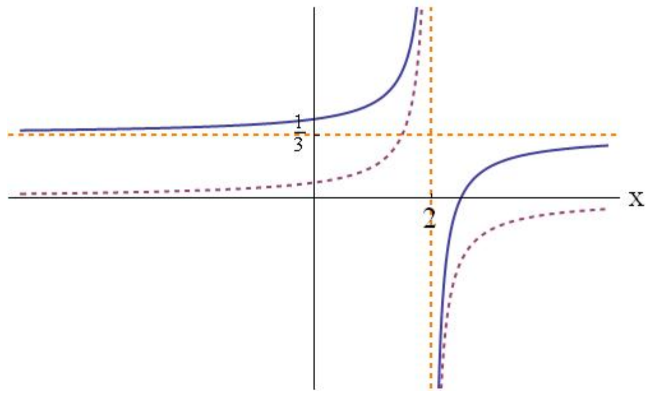
So $f(x) = \dfrac{2x-5}{6x-12} = \dfrac{1}{3} - \dfrac{1}{6x-12}$. Now, we need to find the transformations of $g(x) = \dfrac{1}{x}$ that lead to $f(x)$:

Basic function: $y = g(x) = \dfrac{1}{x}$.
Compress horizontally by 6 units: $y = g(6x) = \dfrac{1}{6x}$.
Shift right by 2 units: $y = g(6(x-2)) = \dfrac{1}{6x-12}$.
Rotate about the x-axis: $y = -g(6x-12) = -\dfrac{1}{6x-12}$.
Shift up 1/3 units: $y = -g(6x-12) + \dfrac{1}{3} = -\dfrac{1}{6x-12} + \dfrac{1}{3} = f(x)$.
and we have found the transformations of $y = 1/x$ which lead to $f(x)$.

Sketches (I have included the previous sketch (dashed line) in the set of transformations for reference purposes only):



$$y=-\frac{1}{6x-12}+\frac{1}{3}=\frac{2x-5}{6x-12}$$



Recall: Graphing a Polynomial $f(x)$ by Hand

Polynomial functions can be sketched by hand if you do the following:

1) Examine end behaviour (horizontal asymptotes, slant asymptotes),
2) Find any zeros (x-intercepts) (factor the polynomial if possible),
3) Find the y-intercept, which is $f(0)$ (it might be a point of interest).

Graphing a Rational Function $f(x)$ by Hand

Rational functions can be sketched by hand if you do the following:

1) Examine end behaviour (horizontal asymptotes, slant asymptotes),
2) Look for vertical asymptotes (factor the denominator if possible),
3) Find any zeros (x-intercepts) (factor the numerator if possible),
4) Find the y-intercept, which is $f(0)$ (it might be a point of interest).

A note on slant asymptotes: The end behaviour analysis we do by looking for dominant terms only allows us to determine the leading behaviour of any asymptote. For horizontal asymptotes, that is all we need since the asymptote is $y = L$. For slant asymptotes, however, we are only able to determine the slope of the slant asymptote since we have discarded all information about the y-intercept of the the slant asymptote. To determine the actual slant asymptote would require us to do long division of polynomials, and since we are only interested in a rough sketch of the main features of the function $y = f(x)$ this step is often not performed. Knowing the slope of the slant asymptote is sufficient.

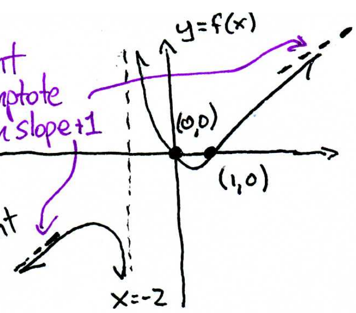
Ex | sketch $f(x)=\frac{x(x-1)}{x+2}$

Zeros: $x=0$, $x=1$ } multiplicity 1, so f changes sign

vertical asymptotes: $x=-2$ multiplicity 1, so f changes sign.

End behaviour: If $|x|$ large, $f(x)\sim\frac{x(x)}{x}=x$ slant asymptote with slope +1.

coefficient +1.



To get full equation of slant asymptote:

$f(x)=\frac{x^2-x}{x+2}$

$$\begin{array}{r} x-3 \\ x+2\,\overline{)\,x^2-x+0} \\ \underline{x^2+2x} \\ -3x+0 \\ \underline{-3x-6} \\ 6 \end{array}$$

$f(x)=x-3+\frac{6}{x+2}$.

If $|x|$ large, $f(x)\sim x-3$

$y=x-3$ is slant asymptote.

Example Sketch the graph of $f(x) = \dfrac{2x-5}{6x-12}$.

The function will have one zero, at $x = 5/2$. Multiplicities:

- zero at $x = 5/2$ has multiplicity 1 (odd) so f changes sign,

The end behaviour is found by determining the leading term in the numerator and denominator, which is

$$\frac{2x-5}{6x-12} \sim \frac{2x}{6x} = \frac{1}{3} \text{ for large } |x|.$$

The function $f(x)$ has a horizontal asymptote $y = 1/3$.

The function will have one vertical asymptote, where $6x - 12 = 0$, at $x = 2$. Multiplicities:

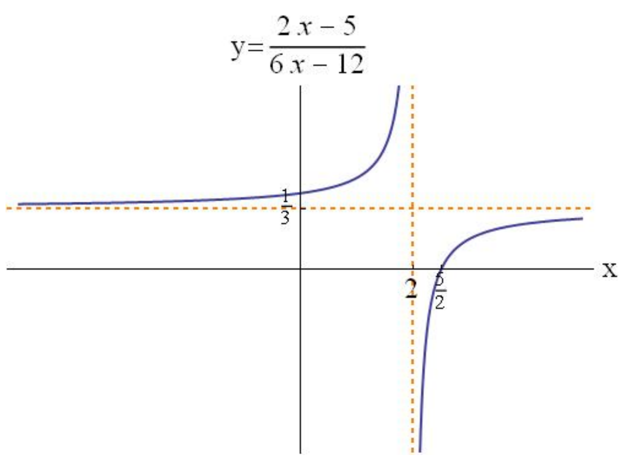
- vertical asymptote at $x = 2$ has multiplicity 1 (odd) so f changes sign.

We now know all the interesting features of the graph!

Sketch by starting on the far right, where we know there is a horizontal asymptote $y = 1/3$.

As you move to the left, the function must cross the x-axis at the zero, $x = 5/2$. It then approaches minus infinity at the vertical asymptote $x = 2$.

The function changes sign at the vertical asymptote, so on the other side it begins up at infinity and drops down to approach the horizontal asymptote $y = 1/3$ again.



I think this was easier than transforming the reciprocal function.

From the sketch, we can, if we wanted to, write things like the following:

$$\begin{aligned} \text{Vertical Asymptote: } & \lim_{x\to 2^+} f(x) = -\infty \\ & \lim_{x\to 2^-} f(x) = +\infty \\ \text{Horizontal Asymptote: } & \lim_{x\to \infty} f(x) = 1/3 \\ & \lim_{x\to -\infty} f(x) = 1/3 \end{aligned}$$

Example Sketch by hand the graph of $f(x) = \dfrac{h(x)}{g(x)} = \dfrac{3x^3 + x - 4}{x^3 + 1}$.

Examine the end behaviour (the leading term is dominant in the numerator and denominator):

$$f(x) = \frac{3x^3 + x - 4}{x^3 + 1} \sim \frac{3x^3}{x^3} = 3 \text{ if } |x| \text{ is large.}$$

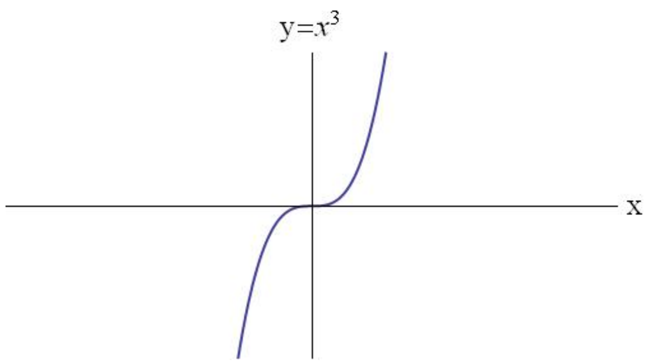
The function $f(x)$ has a horizontal asymptote $y = 3$, since $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to-\infty} f(x) = 3$.

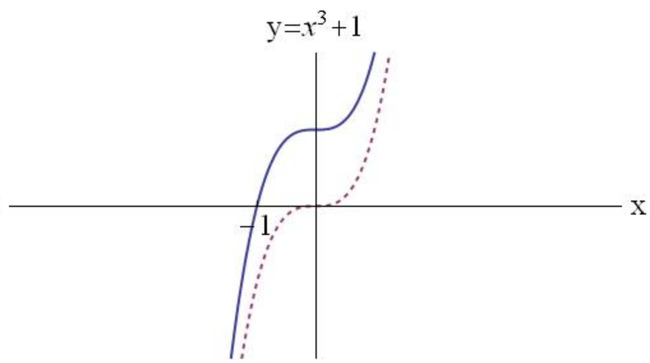
Factor the denominator:

$$g(x) = x^3 + 1$$

This is a sum of cubes, so use the formula for sum of cubes to get $g(x) = x^3 + 1^3 = (x+1)(x^2 - 1\cdot x + 1^2) = (x+1)(x^2 - x + 1)$. If you don't notice this is a sum of cubes (you should), you can always use the ideas from section on Real Zeros of Polynomials to factor.

We know the quadratic $x^2 - x + 1$ will not factor, since we know $g(x) = x^3 + 1$ is simply the cubing function shifted up 1 unit, and so it will only have one x-intercept. Alternately, we could check that $b^2 - 4ac = (-1)^2 - 4(1)(1) = -3 < 0$ so the quadratic has no real roots.





We now need to factor the numerator:

$$h(x) = 3x^3 + x - 4$$

We see by inspection that $h(1) = 0$, so $h(x)$ has a root $x = 1$. Let's factor $(x - 1)$ out:

$$\begin{array}{r|l} & 3x^2 + 3x + 4 \\ x - 1 & 3x^3 + 0x^2 + x - 4 \\ & \underline{3x^3 - 3x^2} \\ & \quad 3x^2 + x - 4 \\ & \quad \underline{3x^2 - 3x} \\ & \qquad 4x - 4 \\ & \qquad \underline{4x - 4} \\ & \qquad\quad 0 \end{array}$$

$$h(x) = 3x^3 + x - 4 = (x - 1)(3x^2 + 3x + 4)$$

The quadratic in this case has $b^2 - 4ac = (3)^2 - 4(3)(4) = -39 < 0$ so the quadratic has no real roots.

What we've learned from the factoring is that we can write

$$f(x) = \frac{h(x)}{g(x)} = \frac{3x^3 + x - 4}{x^3 + 1} = \frac{(x - 1)(3x^2 + 3x + 4)}{(x + 1)(x^2 - x + 1)}$$

The function $f(x)$ has an x-intercept at $x = 1$, since $f(1) = 0$. Since this root is odd multiplicity (1), f will change sign at $x = 1$.

There is a vertical asymptote at $x = -1$. Since this vertical asymptote has odd multiplicity (1), f will f will change sign at $x = -1$.

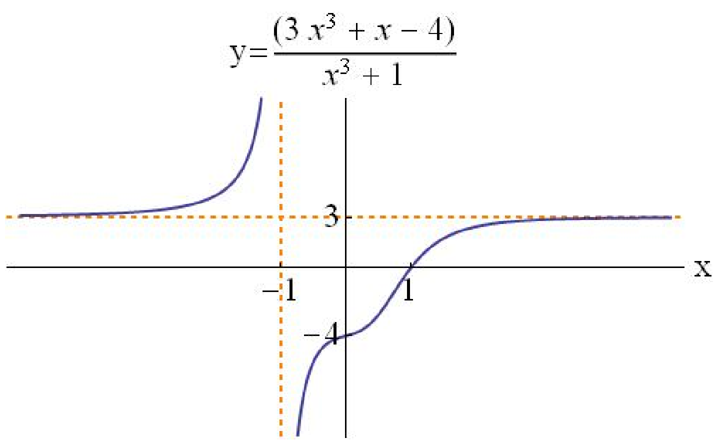
The y-intercept is $f(0) = \dfrac{3(0)^3 + (0) - 4}{(0)^3 + 1} = -4$.

We now know all the interesting features of the graph!

Sketch by starting on the far right, where we know there is a horizontal asymptote $y = 3$.

As you move to the left, the function must cross the x-axis at the zero, $x = 1$. It then passes through the y-intercept and approaches minus infinity at the vertical asymptote $x = -1$.

The function changes sign at the vertical asymptote, so on the other side it begins up at infinity and drops down to approach the horizontal asymptote $y = 3$ again.



From the sketch, we can write things like the following:

$$\text{Vertical Asymptote: } \lim_{x \to -1^+} f(x) = -\infty$$
$$\lim_{x \to -1^-} f(x) = +\infty$$
$$\text{Horizontal Asymptote: } \lim_{x \to \infty} f(x) = 3$$
$$\lim_{x \to -\infty} f(x) = 3$$