Concepts: Long Division Algorithm, The Remainder Theorem, The Rational Zeros Theorem.

This Section looks at some techniques we can use to help us with factoring polynomials.

- Polynomial Division
- The Remainder Theorem
- Rational Zero Theorem

Fundamental Truths for Polynomial Functions All these statements are equivalent. If one is true, all the others are true as well.

1) $x = k$ is a root of the equation $f(x) = 0$.
2) k is a zero of the function f.
3) k is an x-intercept of the graph of $y = f(x)$.
4) $x - k$ is a factor of $f(x)$.

Long Division Algorithm for Polynomials Let f (dividend) and d (divisor) be polynomials with the degree of f greater than the degree of d, and $d(x) \neq 0$. Then there are unique polynomials q (quotient) and r (remainder) such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or the degree of r is less than the degree of d.

Another way of writing this is $\dfrac{f(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)}$.

Synthetic division is another alternative to long division of polynomials. I will not require you to learn synthetic division, only long division of polynomials. Synthetic Division only works if you are dividing by $x - c$, and sometimes you may want to divide by something like $x^2 - 4$, which long division does nicely.

Example Simplify $\dfrac{x^3 - x^2 + 2x - 1}{x + 3}$ so the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator.

$$\begin{array}{r} x^2 - 4x + 14 \\ x+3 \overline{\smash{\big)}\, x^3 - x^2 + 2x - 1} \\ \underline{x^3 + 3x^2} \qquad\qquad \\ -4x^2 + 2x - 1 \\ \underline{-4x^2 - 12x} \quad\; \\ 14x - 1 \\ \underline{14x + 42} \\ -43 \end{array}$$

This means $\dfrac{x^3 - x^2 + 2x - 1}{x + 3} = x^2 - 4x + 14 - \dfrac{43}{x + 3}$

You can check this by find a common denominator:

$$\begin{aligned} & x^2 - 4x + 14 - \frac{43}{x+3} \\ &= \frac{(x^2 - 4x + 14)(x + 3)}{x + 3} - \frac{43}{x + 3} \\ &= \frac{(x^2 - 4x + 14)(x + 3) - 42}{x + 3} \\ &= \frac{x(x^2 - 4x + 14) + 3(x^2 - 4x + 14) - 43}{x + 3} \\ &= \frac{(x^3 - 4x^2 + 14x) + 3x^2 - 12x + 42 - 43}{x + 3} \\ &= \frac{x^3 - x^2 + 2x - 1}{x + 3} \end{aligned}$$

This is useful for factoring since we have shown $x^3 - x^2 + 2x - 1 = (x^2 - 4x + 14)(x + 3) - 43$.

The Remainder Theorem: If a polynomial f is divided by $x - k$, then the remainder is $f(k)$. This means that if $f(k) = 0$, then $x = k$ is a root of f.

Rational Zeros Theorem Suppose f is a polynomial of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with every coefficient an integer and $a_0 \neq 0$. If $x = p/q$ is a rational zero of f, where p and q have no common integer factors other than 1, then

1) p is an integer factor of the the constant coefficient a_0.
2) q is an integer factor of the leading coefficient a_n.

These two theorem gives us a place to start when faced with a polynomial that requires factoring.

Example Find all of the real zeros of the function $f(x) = 2x^3 - 3x^2 - 4x + 6$.

Since the coefficients are all integers, we will use the Rational Zero Theorem to get us started.
Factors of $a_0 = 6$: $\pm1, \pm2, \pm3, \pm6$.
Factors of $a_4 = 2$: $\pm1, \pm2$.
Potential rational zeros: $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$.

$$\begin{aligned}
f(1) &= 2(1)^3 - 3(1)^2 - 4(1) + 6 = +1 \neq 0 \\
f(1/2) &= 2(1/2)^3 - 3(1/2)^2 - 4(1/2) + 6 = 7/2 \neq 0 \\
f(3/2) &= 2(3/2)^3 - 3(3/2)^2 - 4(3/2) + 6 = 0
\end{aligned}$$

Therefore, $x = 3/2$ is a root. Let's factor it out using long division.

$$\begin{array}{r|l}
 & 2x^2 - 4 \\
\hline
x - \frac{3}{2} & 2x^3 - 3x^2 - 4x + 6 \\
 & 2x^3 - 3x^2 \\
\hline
 & \quad\quad -4x + 6 \\
 & \quad\quad -4x + 6 \\
\hline
 & \quad\quad\quad 0
\end{array}$$

$$f(x) = 2x^3 - 3x^2 - 4x + 6 = \left(x - \frac{3}{2}\right)(2x^2 - 4) = (2x - 3)(x^2 - 2) = (2x - 3)(x + \sqrt{2})(x - \sqrt{2})$$

There are three real roots, one rational ($x = 3/2$) and two irrational ($x = \pm\sqrt{2}$). Each root has multiplicity one.

Note: This question could be asked on a test in the following manner:
Find all of the real zeros of the function $f(x) = 2x^3 - 3x^2 - 4x + 6$ given $x = 3/2$ is a zero of f.

Example Sketch by hand the function $f(x) = 2x^4 - 11x^3 + 22x^2 - 19x + 6$, given $x = 1$ is a root of multiplicity 2 of f. Since $x = 1$ is a root, we can factor it out using long division.

$$2x^3 - 9x^2 + 13x - 6$$
$$x-1 \,\big)\, 2x^4 - 11x^3 + 22x^2 - 19x + 6$$
$$2x^4 - 2x^3$$
$$-9x^3 + 22x^2 - 19x + 6$$
$$-9x^3 + 9x^2$$
$$13x^2 - 19x + 6$$
$$13x^2 - 13x$$
$$-6x + 6$$
$$-6x + 6$$
$$0$$

$$2x^2 - 7x + 6$$
$$x-1 \,\big)\, 2x^3 - 9x^2 + 13x - 6$$
$$2x^3 - 2x^2$$
$$-7x^2 + 13x - 6$$
$$-7x^2 + 7x$$
$$6x - 6$$
$$6x - 6$$
$$0$$

$$f(x) = 2x^4 - 11x^3 + 22x^2 - 19x + 6 = (x-1)(2x^3 - 9x^2 + 13x - 6) = (x-1)g(x)$$

Since $x = 1$ is a root of multiplicity 2, it can be factored out of $g(x)$! Let's factor it out using long division (details above.

$$g(x) = 2x^3 - 9x^2 + 13x - 6 = (x-1)(2x^2 - 7x + 6)$$

$$f(x) = (x-1)g(x) = (x-1)^2(2x^2 - 7x + 6)$$

We can do the factoring of the quadratic by inspection, $f(x) = (x-1)^2(2x-3)(x-2)$. Now we can sketch, since the factoring is done.

The polynomial will have three zeros, at $x = 1, 3/2, 2$. Multiplicities:

- zero at $x = 1$ has multiplicity 2 (even) so f does not change sign,
- zero at $x = 3/2$ has multiplicity 1 (odd) so f changes sign,
- zero at $x = 2$ has multiplicity 1 (odd) so f changes sign.

The end behaviour of the polynomial is found by determining the leading term, which is

$$(x-1)^2(2x-3)(x-2) \sim (x)^2(2x)(x) = 2x^4 \text{ for large } |x|.$$

The end behaviour of the monomial $2x^4$ is $\lim_{x\to-\infty} 2x^4 = \infty \qquad \lim_{x\to\infty} 2x^4 = \infty$.