

Concepts: graphing inequalities in two variables, regions, linear programming (constraints, feasible region, corner points, objective function).

In this section we will look at solving inequalities by graphing and identifying the region that satisfies all the inequalities. In calculus, this skill will be useful if you need to determine the region enclosed by $y = f(x)$ and $y = g(x)$, which is a first step when performing volumes of rotation (in Calculus II).

If we stick to linear inequalities, we are led not to a calculus application but instead to the topic *Linear Programming*, where we are interested in optimizing (finding max or min) a given function subject to a variety of constraints (the inequalities). In this situation, the region that satisfies all the constraints is called the *feasible region*.

We will get a little insight into what Linear Programming is and how it works, but there is a vast array of mathematics built up around linear programming. It is one of the most important mathematical applications, so it is nice to see a little bit of it here.

Steps to Solving System of Inequalities Graphically

Consider the system of inequalities (note we could have $\leq, \geq, <, >$ in each case):

$$F(x, y) < 0$$

$$G(x, y) > 0$$

For $F(x, y) < 0$:

1. Sketch the equality $F(x, y) = 0$. Use dashed line if you have $<$ or $>$ and solid line if you have \leq, \geq . The dashed line indicates that the boundary is not part of the set of points that satisfies the inequality.
2. Pick a test point not on the line, and see if the inequality is true at that point. If it is, shade the side of the curve that contains the test point. If it is false, shade the side of the curve that does not contain the test point.

Repeat the procedure for the other inequality.

Identify the region that satisfies all the inequalities.

If asked, determine any points of intersection where the boundary of the region changes from one curve to another. This might be very difficult or even impossible to do!

Example Graph by hand the solution to the system of inequalities, and determine the points of intersection for the region. Finally, describe the region mathematically.

$$x - 3y - 6 \leq 0$$

$$y + x^2 + 2x \geq 2$$

Sketch $x - 3y - 6 = 0$

If $x=0$, then $-3y - 6 = 0$
 $y = -2$

$$\Rightarrow (0, -2)$$

If $y=0$, then $x - 6 = 0$
 $x = 6$

$$\Rightarrow (6, 0)$$

Test point $(0,0)$:

$$(0) - 3(0) - 6 \leq 0$$

$$-6 \leq 0 \text{ True!}$$

shade the side of line containing $(0,0)$.

Sketch $y + x^2 + 2x = 2$
 quadratic. Complete square.

$$y = -x^2 - 2x + 2$$

$$= -(x^2 + 2x - 2)$$

$$= -(x^2 + 2x + 1 - 1 - 2)$$

$$= -([x+1]^2 - 3)$$

$$= -[x+1]^2 + 3$$

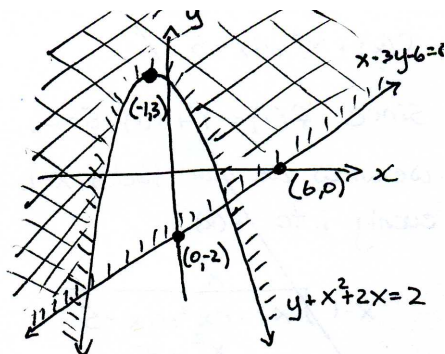
opens down, vertex $(-1, 3)$

Test point $(0,0)$:

$$(0) + (0)^2 + 2(0) \geq 2$$

$$0 \geq 2 \text{ False!}$$

shade side of curve not containing $(0,0)$.



The region is the cross hatched area. To determine the points of intersection, solve

$$x - 3y - 6 = 0$$

$$y + x^2 + 2x = 2$$

Take the first equation and solve for x , substitute into the second equation. Then solve for y (details left out):

$$y = \frac{1}{18} (-43 \pm \sqrt{193})$$

Substituting this into the first equation, we get the ordered pairs for the points of intersection:

$$\left(\frac{1}{6} (-7 - \sqrt{193}), \frac{1}{18} (-43 - \sqrt{193}) \right) \text{ and } \left(\frac{1}{6} (-7 + \sqrt{193}), \frac{1}{18} (-43 + \sqrt{193}) \right)$$

We can now describe the region in words as follows:

The region is above the curve $y + x^2 + 2x = 2$ for $x \in [1/6(-7 - \sqrt{193}), 1/6(-7 + \sqrt{193})]$, and above the curve $x - 3y - 6 = 0$ for all other x .

Aside: *Mathematica* can create plots of inequalities using the command **RegionPlot**.

Linear Programming

Linear Programming is the most used mathematical tool in management science. It is used to maximize (or minimize) a quantity based on constraints. Let's examine Linear Programming by working through a specific example.

Example A clothing manufacturer has 600 yds of cloth available to make shirts and decorated vests. Each shirt requires 3 yds of material and yields a profit of \$5. Each vest requires 2 yds of material and yields a profit of \$2. The manufacturer wants to guarantee that at least 100 shirts and 30 vests are produced, to keep the product line diversified. How many of each garment should be made to maximize profit?

To begin, we need to introduce some notation. Let x be the number of vest made, and y the number of shirts made.

A mixture chart simply writes all the information provided in a convenient chart, with the products as columns down the right, and the resources as rows. In this case,

	Cloth Required (600 yards available)	Minimum that must be made	Profit
number of vests made, x	2 yards	30	\$2
number of shirts made, y	3 yards	100	\$5

Now we need to translate all of the information in the mixture chart into some mathematical formulas.

There is a limit to the number of shirts and vests we can make due to the amount of cloth available. This leads to the constraint:

$$2x + 3y \leq 600.$$

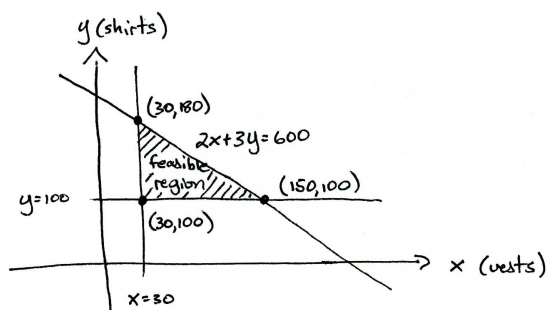
The manufacturer also imposed some constraints on the minimum number of garments that can be made, which leads to the constraints:

$$x \geq 30 \text{ and } y \geq 100.$$

Therefore, the only constraints we have are that

$$2x + 3y \leq 600, x \geq 30 \text{ and } y \geq 100.$$

These constraints give rise to the feasible region. Sketching the region which satisfies the constraints in the xy -plane will tell us what number of shirts and vests it is possible (feasible) for the company to manufacture, given those constraints. That is, it represents all the physically possible solutions to the problem.



So any point in the shaded region is a production schedule the company could use. For example, they could choose to produce $x = 100$ vests and $y = 125$ shirts. We now need to answer the question: Which point in the feasible region produces the maximum profit?

The profit (sometimes called the *objective function*) can be found from the mixture chart we created earlier. The profit P (in dollars) is

$$P = 2x + 5y$$

So we can work out the profit if we know values of x and y . For example, the profit of producing $x = 100$ vests and $y = 125$ shirts is

$$P = 2(100) + 5(125) = \$825.$$

But is this a maximum? Probably not.

The corner point principle tells us that the maximum solution will occur at one of the corners of the feasible region.

Determine the corner points: $(30,100)$, $(300/2,100)=(150,100)$, and $(30, 540/3)=(30,180)$.

So all we need to do is calculate the profit at all the corner points,

Corner Point	Profit
$x = 30, y = 100$	$P = 2(30) + 5(100) = \$560$
$x = 30, y = 180$	$P = 2(30) + 5(180) = \$960$
$x = 150, y = 100$	$P = 2(150) + 5(100) = \$800$

So the optimal production schedule, which will maximize profits, is to produce 30 vests and 180 shirts.

Why the Corner Point Principle Works

The corner point principle works, since the profit is a straight line. For different value of P , we get different straight lines. As these straight lines sweeps over the feasible region (as profit P increases), you will always end with the profit line touching a corner of the feasible region! You might end up with the maximum profit line touching two corner points, which can happen if the profit line is parallel to one of the constraints. In this situation, you would have more than one solution that maximized the profit.

There is an animation of this online.

Beyond Linear Programming

There is such a thing as integer programming, when all the numbers must be integers. This makes sense, since what if we had found that we needed to produce 124.5 shirts and 30 vests? How can we produce half a shirt?

The problem is integer programming is much harder to solve than linear programming! It is not unusual to use linear programming as an approximation of a problem really should be done using integer programming, and then rounding your answers to integers.