Questions

1. Factor $f(x) = x^4 + x^3 + 5x^2 - x - 6$.

2. Factor $f(x) = x^3 + 4x - 5$.

3. Determine the real and imaginary parts of the complex number $\sqrt{i}$. *Hint:* Let $\sqrt{i} = a + bi$ $a, b \in \mathbb{R}$, and then try to determine the values of a and b.

4. Verify that $-2i$ is a zero of $f(x) = x^3 - (2-i)x^2 + (2-2i)x - 4$.

5. If z is a complex number, which of the following is a real number?
(a) $z + \bar{z}$
(b) $z\bar{z}$
(c) $(z + \bar{z})^2$
(d) $(z\bar{z})^2$
(e) z^2

6. Write the polynomial $f(x) = x(x-1)(x-1-i)(x-1+i)$ in standard form, and identify the zeros of the function and the x-intercepts of its graph.

7. Can a polynomial $f(x)$ of degree five with real coefficients have exactly two x-intercepts? Explain.

Solutions

1. Factor $f(x) = x^4 + x^3 + 5x^2 - x - 6$.

$f(x) = x^4 + x^3 + 5x^2 - x - 6$

Notice $f(1) = 0$.

Therefore, divide $x-1$ into $f(x)$.

$$
\begin{array}{r}
x^3 + 2x^2 + 7x + 6 \\
x-1 \,\overline{\big)\, x^4 + x^3 + 5x^2 - x - 6} \\
\underline{x^4 - x^3 \qquad\qquad\qquad} \\
2x^3 + 5x^2 - x - 6 \\
\underline{2x^3 - 2x^2 \qquad\quad} \\
7x^2 - x - 6 \\
\underline{7x^2 - 7x \quad\;\;} \\
6x - 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

$f(x) = (x-1)(x^3 + 2x^2 + 7x + 6)$

Note $x^3 + 2x^2 + 7x + 6 \big|_{x=-1} = 0$

so divide $x+1$ into $x^3 + 2x^2 + 7x + 6$.

$$
\begin{array}{r}
x^2 + x + 6 \\
x+1 \,\overline{\big)\, x^3 + 2x^2 + 7x + 6} \\
\underline{x^3 + x^2 \qquad\qquad} \\
x^2 + 7x + 6 \\
\underline{x^2 + x \quad\;\;} \\
6x + 6 \\
\underline{6x + 6} \\
0
\end{array}
$$

$f(x) = (x-1)(x+1)(x^2 + x + 6)$.

Quadratic formula to get zeros of quadratic:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(6)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{23}}{2} i$$

Zeros of $f(x)$ are $x = \pm 1$, $x = -\frac{1}{2} \pm \frac{\sqrt{23}}{2} i$.

Factored form:

$$f(x) = (x-1)(x+1)\left(x - \left[-\tfrac{1}{2} + \tfrac{\sqrt{23}}{2} i\right]\right)\left(x - \left[-\tfrac{1}{2} - \tfrac{\sqrt{23}}{2} i\right]\right)$$

$$= (x-1)(x+1)\left(x + \tfrac{1}{2} - \tfrac{\sqrt{23}}{2} i\right)\left(x + \tfrac{1}{2} + \tfrac{\sqrt{23}}{2} i\right)$$

2. Factor $f(x) = x^3 + 4x - 5$.

$f(x) = x^3 + 4x - 5$

Since $f(1) = 1^3 + 4(1) - 5 = 0$, we know we can factor $x-1$ evenly into $f(x)$.

$$
\begin{array}{r}
x^2 + x + 5 \\
x-1 \,\overline{\big)\, x^3 + 0x^2 + 4x - 5} \\
\underline{x^3 - x^2 \qquad\qquad} \\
x^2 + 4x - 5 \\
\underline{x^2 - x \quad\;\;} \\
5x - 5 \\
\underline{5x - 5} \\
0
\end{array}
$$

$f(x) = (x-1)(x^2 + x + 5)$

Use quadratic formula to factor quadratic, $x^2 + x + 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-19}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{19}}{2} i$$

Zeros are $x = 1, -\frac{1}{2} + \frac{\sqrt{19}}{2} i, -\frac{1}{2} - \frac{\sqrt{19}}{2} i$.

Factors are $x - 1$

$x - \left(-\frac{1}{2} + \frac{\sqrt{19}}{2} i\right)$

$x - \left(-\frac{1}{2} - \frac{\sqrt{19}}{2} i\right)$

$$f(x) = (x-1)\left(x + \tfrac{1}{2} - \tfrac{\sqrt{19}}{2} i\right)\left(x + \tfrac{1}{2} + \tfrac{\sqrt{19}}{2} i\right)$$

3. Determine the real and imaginary parts of the complex number $\sqrt{i}$. *Hint:* Let $\sqrt{i} = a + bi$ $a, b \in \mathbb{R}$, and then try to determine the values of a and b.

Let $\sqrt{i} = a+bi$, $a,b \in \mathbb{R}$.
and we need to determine a and b.

Square both sides:

$i = (a+bi)^2$

$i = a^2 + 2abi + b^2 i^2$ use $i^2 = -1$.

$i = a^2 + 2abi - b^2$

$i = a^2 - b^2 + 2abi$

Equate real parts: $0 = a^2 - b^2$

Equate complex parts: $i = 2abi$

$\Rightarrow 1 = 2ab$.

Solve $\left.\begin{matrix} 0 = a^2 - b^2 \\ 1 = 2ab \end{matrix}\right\}$ 2 equations in 2 unknowns a and b.

First equation: $a^2 = b^2$

$a = \pm b$.

Second equation $\cancel{b = \frac{1}{2a}}$. $1 = 2ab$:

If $a = b$: $1 = 2(b)b \Rightarrow b^2 = \frac{1}{2} \Rightarrow b = \pm \frac{1}{\sqrt{2}}$.

so $a = \frac{1}{\sqrt{2}}$ $b = \frac{1}{\sqrt{2}}$ or $a = -\frac{1}{\sqrt{2}}$ $b = -\frac{1}{\sqrt{2}}$

are two sets of solutions.

If $a = -b$: $1 = 2(-b)b \Rightarrow b^2 = \frac{-1}{2}$. no solution since we assumed $b \in \mathbb{R}$.

Therefore $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$.

4. Verify that $-2i$ is a zero of $f(x) = x^3 - (2-i)x^2 + (2-2i)x - 4$.

$$
\begin{aligned}
f(x) &= x^3 - (2-i)x^2 + (2-2i)x - 4 \\
f(-2i) &= (-2i)^3 - (2-i)(-2i)^2 + (2-2i)(-2i) - 4 \\
&= (-2)^3 i^3 - (2-i)(-2)^2 i^2 - 4i + 4i^2 - 4 \\
&= (-8)(i^2)i - (2-i)(4)(-1) - 4i + 4(-1) - 4 \\
&= (-8)(-1)i - (-8+4i) - 4i - 8 \\
&= 8i + 8 - 4i - 4i - 8 \\
&= 0
\end{aligned}
$$

5. If z is a nonreal complex number, which of the following is a real number?
(a) $z + \bar{z}$
(b) $z\bar{z}$
(c) $(z + \bar{z})^2$
(d) $(z\bar{z})^2$
(e) z^2

Let $a = a + bi$ with $a, b \in \mathbb{R}$, so $\bar{z} = a - bi$ and simplify each case.

(a) $z + \bar{z} = a + bi + a - bi = 2a \in \mathbb{R}$

(b) $z\bar{z} = (a+bi)(a-bi) = a^2 - bi^2 = a^2 + b^2 \in \mathbb{R}$

(c) $(z+\bar{z})^2 \in \mathbb{R}$ since $z + \bar{z} \in \mathbb{R}$

(d) $(z\bar{z})^2 \in \mathbb{R}$ since $z\bar{z} \in \mathbb{R}$

(e) $z^2 = (a+bi)^2 = a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi$ is not $\mathbb{R}$ unless $a = 0$

6. Write the polynomial $f(x) = x(x-1)(x-1-i)(x-1+i)$ in standard form, and identify the zeros of the function and the x-intercepts of its graph.

Zeros of $f(x)$ are: $x = 0, 1, 1 \pm i$.

x-intercepts of the graph of $f(x)$ are the real zeros only, so $x = 0, 1$.

Multiply out to get in standard form:

$$\begin{aligned}
f(x) &= x(x-1)(x-1-i)(x-1+i) \\
&= (x^2-x)(x-[1+i])(x-[1-i]) \\
&= (x^2-x)(x^2+[1+i][1-i]-x[1+i]-x[1-i]) \\
&= (x^2-x)(x^2+1-i^2-x\cancel{-xi}-x\cancel{+xi}) \\
&= (x^2-x)(x^2+1-(-1)-2x) \\
&= (x^2-x)(x^2+2-2x) \\
&= (x^2-x)(x^2-2x+2) \\
&= x^2(x^2-2x+2)-x(x^2-2x+2) \\
&= x^4-2x^3+2x^2-x^3+2x^2-2x \\
&= x^4-3x^3+4x^2-2x
\end{aligned}$$

7. Can a polynomial $f(x)$ of degree five with real coefficients have exactly two x-intercepts? Explain.

Yes, but one of the x-intercepts must be of multiplicity two or four so the function $f(x)$ does not change sign there. The other x-intercept will be of multiplicity one. For a specific example, consider $f(x) = (x-1)x^2(x^2+1) = x^5 - x^4 + x^3 - x^2$, which is degree five with x-intercepts of 0 (multiplicity 2) and 1 (multiplicity 2). An example having multiplicity four for one of the intercepts would be $f(x) = (x-1)x^4$.

Note that x-intercepts correspond to real valued factors.