Questions

1. Sketch $6x + 3y - x^2 = 9$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

2. Sketch $3y^2 - 4y + 3x - 7 = 0$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

3. Sketch $y^2 - 3y - 3x + 7 = 0$ and $y - x^2 + x = 0$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

4. Analyze the quadratic $y = ax^2 + bx + c$, $a > 0$, and show that it is a parabola. Determine the vertex, focus, and directrix.

Solutions

1. Sketch $6x + 3y - x^2 = 9$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

$6x + 3y - x^2 = 9$

Complete the square in x.

$-1(x^2 - 6x) = -3y + 9$

$-1(\underbrace{x^2 - 6x + 9} - 9) = -3y + 9$

$-1([x-3]^2 - 9) = -3y + 9$

$-[x-3]^2 + \cancel{9} = -3y + \cancel{9}$

$(x-3)^2 = 3y$

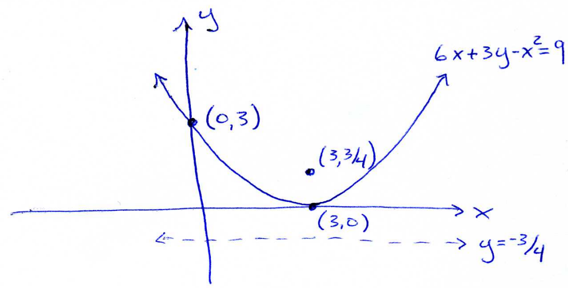
This is standard form of parabola opening up, with ~~center~~ vertex $(3,0)$.

$(x-h)^2 = 4p(y-k)$

Also, $4p = 3 \Rightarrow p = 3/4$.

Focus is $(3, 3/4) = (h, k+p)$

Directrix is $y = -3/4$.



2. Sketch $3y^2 - 4y + 3x - 7 = 0$ by hand. Include all steps in your solution. Identify the focus and directrix of the parabola.

$3y^2 - 4y + 3x - 7 = 0$

complete square in y

$3\left[\underbrace{y^2 - \frac{4}{3}y + \left(\frac{2}{3}\right)^2}_{} - \left(\frac{2}{3}\right)^2\right] + 3x - 7 = 0$

$3\left[\left(y - \frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\right] + 3x - 7 = 0$

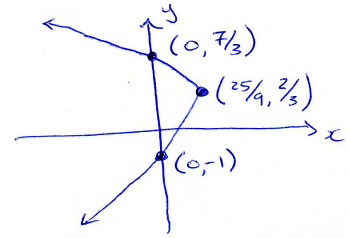
$3\left(y - \frac{2}{3}\right)^2 - \frac{4}{3} + 3x - 7 = 0$

$3\left(y - \frac{2}{3}\right)^2 = -3x + \frac{25}{3}$

$\left(y - \frac{2}{3}\right)^2 = -x + \frac{25}{9}$

$\left(y - \frac{2}{3}\right)^2 = -1\left(x - \frac{25}{9}\right)$

This is the standard form of a parabola opening to the left with vertex $\left(\frac{25}{9}, \frac{2}{3}\right)$.



If $x=0$, $\left(y - 2/3\right)^2 = 25/9$

$y = \frac{2}{3} \pm \frac{5}{3} = \frac{7}{3}, -1.$

I wanted to get these points $(0, 7/3)$ and $(0,-1)$, to make my sketch more accurate.

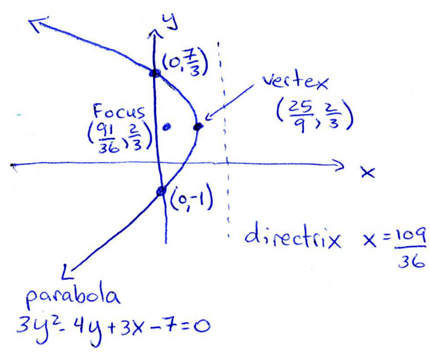
The standard form is

$(y-k)^2 = -4p(x-h)$

so $-4p = -1 \Rightarrow p = 1/4$.

The focus is $\left(\frac{25}{9} - \frac{1}{4}, 2/3\right) = \left(\frac{91}{36}, 2/3\right)$

The directrix is $x = \frac{25}{9} + \frac{1}{4} = \frac{109}{36}$



3. Sketch $y^2 - 3y - 3x + 7 = 0$ and $y - x^2 + x = 0$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

Sketch $y^2 - 3y - 3x + 7 = 0$

$$\underbrace{y^2 - 3y + \left(\tfrac{3}{2}\right)^2} - \left(\tfrac{3}{2}\right)^2 = 3x - 7$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3x - 7 + \frac{9}{4}$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3x - \frac{19}{4}$$

$$\left(y - \tfrac{3}{2}\right)^2 = 3\left(x - \frac{19}{12}\right)$$

parabola, opens right, vertex $(19/12, 3/2)$.

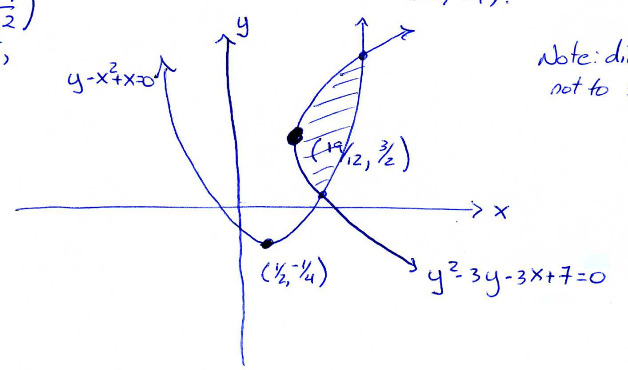
Sketch $y - x^2 + x = 0$

$$x^2 - x = y$$

$$\underbrace{x^2 - x + \tfrac{1}{4}} - \frac{1}{4} = y$$

$$\left(x - \tfrac{1}{2}\right)^2 = y + \frac{1}{4}$$

parabola, opens up, vertex $(1/2, -1/4)$.



Note: diagram is not to scale.

Points of intersection:

$y - x^2 + x = 0 \rightarrow y = x^2 - x$, sub into other equation

$(x^2 - x)^2 - 3(x^2 - x) - 3x + 7 = 0$ which will be difficult to solve.

We would need to use a computer to proceed.

4. Analyze the quadratic $y = ax^2 + bx + c$, $a > 0$, and show that it is a parabola. Determine the vertex, focus, and directrix.

$$y = ax^2 + bx + c$$

$$y - c = a\left(x^2 + \frac{b}{a}x\right)$$

$$= a\left[\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

$$y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right)$$

Compare to $(x-h)^2 = 4p(y-k)$

parabola opens up, vertex (h,k).

So if $a > 0$, then this is a parabola that opens up with vertex $\left(-\frac{b}{2a}, -\frac{(b^2-4ac)}{4a}\right)$.

Since $4p = \frac{1}{a} \Rightarrow p = \frac{1}{4a}$.

Focus $(h, k+p) = \left(\frac{-b}{2a}, -\frac{(b^2-4ac)}{4a} + \frac{1}{4a}\right)$

~~Directrix~~ $\cancel{(h, k-p) = \left(\frac{-b}{2a}, -\frac{(b^2-4ac)}{4a} - \frac{1}{4a}\right)}$

Aside, earlier in the course, we saw the vertex of $f(x) = ax^2 + bx + c$ was $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Since $f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$

$$= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}$$

$$= \frac{-b^2 + 4ac}{4a} = -\frac{(b^2-4ac)}{4a}$$

this agrees with what we found here.

Whoops! I wasn't thinking.

The directrix is $y = k - p$

$$y = -\frac{(b^2-4ac)}{4a} - \frac{1}{4a}$$

$$y = \frac{-1 - b^2 + 4ac}{4a}$$