Questions

1. Sketch $9x^2 + 4y^2 - 18x + 8y - 23 = 0$ by hand. Include all steps in your solution. Identify the vertices and foci of the ellipse.

2. Sketch $x^2/4 + y^2/9 = 1$ and $x^2 + y^2 = 4$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

3. The graph of the equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 0$ is considered to be a degenerate ellipse. Describe the graph.

4. Prove that the nondegenerate graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is an ellipse if $AC > 0$.

5. Determine the perimeter of a triangle with one vertex on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the other two vertices at the foci of the ellipse.

Solutions

1. Sketch $9x^2 + 4y^2 - 18x + 8y - 23 = 0$ by hand. Include all steps in your solution. Identify the vertices and foci of the ellipse.

$9x^2 + 4y^2 - 18x + 8y - 23 = 0$

complete the square in both x and y.

$9x^2 - 18x + 4y^2 + 8y = 23$

$9[\underbrace{x^2 - 2x + 1} - 1] + 4[\underbrace{y^2 + 2y + 1} - 1] = 23$

$9[(x-1)^2 - 1] + 4[(y+1)^2 - 1] = 23$

$9(x-1)^2 - 9 + 4(y+1)^2 - 4 = 23$

$9(x-1)^2 + 4(y+1)^2 = 36$

$\dfrac{(x-1)^2}{2^2} + \dfrac{(y+1)^2}{3^2} = 1$

ellipse! center: $(1,-1)$.

If $x = 1$, $\dfrac{(y+1)^2}{3^2} = 1$

$y + 1 = \pm 3$

$y = 2, -4$.

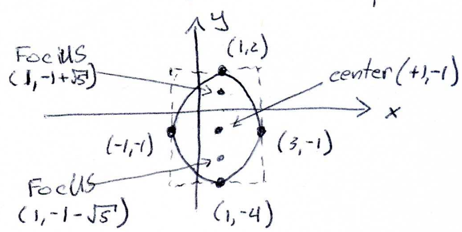
so $(1,2)$, $(1,-4)$ are on ellipse.

If $y = -1$, $\dfrac{(x-1)^2}{2^2} = 1$

$x - 1 = \pm 2$

$x = 3, -1$

so $(-1,-1)$, $(3,-1)$ are on ellipse.



Since $c^2 = 3^2 - 2^2 = 5$, $c = \sqrt{5}$.

Vertices are ends of ellipse along stretched direction (major axis), which are $(1,2)$, $(1,-4)$.

2. Sketch $x^2/4 + y^2/9 = 1$ and $x^2 + y^2 = 4$ by hand on the same set of axis. Do the curves intersect? If so, can you determine the points of intersection by hand?

sketch $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

ellipse → get the box that bounds it!

If $x = 0$, $\dfrac{y^2}{9} = 1$

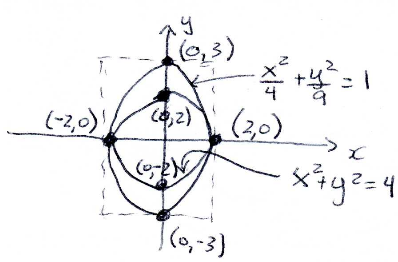
$y = \pm 3$

$(0,3)$, $(0,-3)$ on box

If $y = 0$, $\dfrac{x^2}{4} = 1$

$x = \pm 2$

$(2,0)$, $(-2,0)$ on box



sketch $x^2 + y^2 = 4$

$x^2 + y^2 = 2^2$

circle, center $(0,0)$ radius 2.

Verify intersection is $(\pm 2, 0)$:

Solve $\left.\begin{array}{l} \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \\ x^2 + y^2 = 4 \end{array}\right\}$ system of 2 equations in 2 unknowns.

From second equation, $x^2 = 4 - y^2$, sub into first equation:

$\dfrac{4 - y^2}{4} + \dfrac{y^2}{9} = 1$

$1 - \dfrac{y^2}{4} + \dfrac{y^2}{9} = 1$

$\rightarrow y = 0$.

→ If $y = 0$, then $x^2 + 0^2 = 4$

$x = \pm 2$.

only two points of intersection, $(2,0)$, and $(-2,0)$.

3. The graph of the equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=0$ is considered to be a degenerate ellipse. Describe the graph.

If $x=h$, then $\dfrac{(y-k)^2}{b^2}=0$

$y=k$.

point (h,k).

If $y=k$, then $\dfrac{(x-h)^2}{a^2}=0$

$x=h$.

point (h,k).

So it looks like $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=0$ consists of a single point, (h,k), the center of the ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$.

You can also see this from trying to solve directly for y:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=0$$

$$(y-k)^2=-\frac{b^2}{a^2}(x-h)^2$$

$$y-k=\pm\sqrt{-\frac{b^2}{a^2}(x-h)^2}$$

$$=\frac{b}{a}(x-h)i$$

$$y=k+\frac{b}{a}(x-h)i.$$

So the only real valued solution is when $x=h$, where $y=k$. The point (h,k)

4. Prove that the nondegenerate graph of the equation $Ax^2+Cy^2+Dx+Ey+F=0$ is an ellipse if $AC>0$.

$Ax^2+Cy^2+Dx+Ey+F=0$

complete the square in x and y.

$$A\Big[\underbrace{x^2+\tfrac{D}{A}x+\left(\tfrac{D}{2A}\right)^2}-\left(\tfrac{D}{2A}\right)^2\Big]+C\Big[\underbrace{y^2+\tfrac{E}{C}y+\left(\tfrac{E}{2C}\right)^2}-\left(\tfrac{E}{2C}\right)^2\Big]=-F$$

$$A\left[\left(x+\tfrac{D}{2A}\right)^2-\left(\tfrac{D}{2A}\right)^2\right]+C\left[\left(y+\tfrac{E}{2C}\right)^2-\left(\tfrac{E}{2C}\right)^2\right]=-F$$

$$A\left(x+\tfrac{D}{2A}\right)^2-\frac{D^2}{4A}+C\left(y+\tfrac{E}{2C}\right)^2-\frac{E^2}{4C}=-F$$

$$A\left(x+\tfrac{D}{2A}\right)^2+C\left(y+\tfrac{E}{2C}\right)^2=\frac{D^2}{4A}+\frac{E^2}{4C}-F$$

$$\frac{\left(x+\frac{D}{2A}\right)^2}{C}+\frac{\left(y+\frac{E}{2C}\right)^2}{A}=\frac{D^2}{4A^2C}+\frac{E^2}{4AC^2}-\frac{F}{AC}$$

$$\frac{\left(x+\frac{D}{2A}\right)^2}{C}+\frac{\left(y+\frac{E}{2C}\right)^2}{A}=\frac{D^2C+E^2A-4ACF}{4A^2C^2}$$

⇑ we need this to be 1 for an ellipse.

What we are concerned about are the signs of the factors. Let $W=\dfrac{D^2C+E^2A-4ACF}{4A^2C^2}$, then

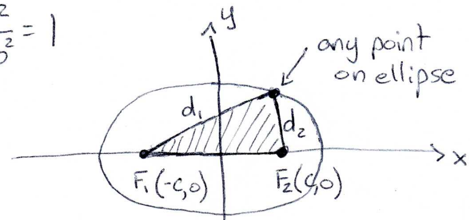
$$\frac{\left(x+\frac{D}{2A}\right)^2}{C\cdot W}+\frac{\left(y+\frac{E}{2C}\right)^2}{A\cdot W}=1$$

If $C>0, A>0, W>0$ this is an ellipse.

Also an ellipse if $C<0, A<0, W<0$.

5. Determine the perimeter of a triangle with one vertex on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the other two vertices at the foci of the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



By definition, $d_1 + d_2 = 2a$.
(see our derivation, we included the 2 to simplify things later)

$F_1F_2 = 2c$.

So Perimeter of triangle is $2a + 2c$.