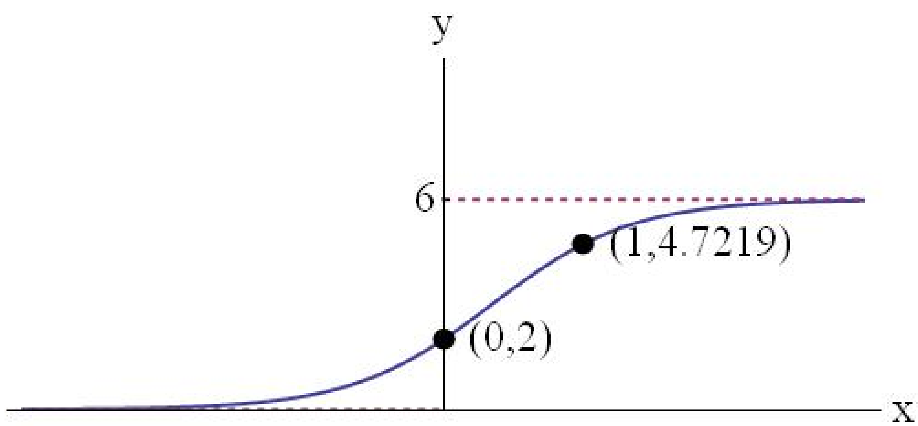
Questions

1. The half-life of a radioactive substance is 10 days and there are 5 g present initially. Find an expression for the amount of material left after t days have elapsed. Show the process you used clearly, and do not begin with a formula. The idea is for you to end with a formula, not begin with it.

Then, find an expression for the time it would take to have m amount of material left. Explain using mathematics and the expression you find for the time that it must be true that $t > 0$.

2. Determine an algebraic formula of the form $f(x) = \dfrac{c}{1 + ae^{-kx}}$ for the logistic function whose graph is shown below.



3. Consider depositing an amount of money $\$P$ into a savings account. After each month, the account earns $\frac{r}{12}\%$ interest based on the amount of money currently in the account. Therefore, r is the interest rate per year. This is called *compound interest* (you are earning interest on interest). Derive a formula for the amount of money in the account after one year.

Modify your formula for the case when interest is paid every day (assume 365 days in a year).

Modify your formula for the case when interest is paid every hour (assume 365 days in a year).

What would you guess the formula would be if interest was paid continuously?

4. A researcher wishes to determine an exponential growth model for mold. She measures the amount of mold at $t = 0$ as 2mm^2. Two days later she finds 4mm^2.

Determine the exponential growth model $P(t) = a \cdot e^{kt}$. How long will the mold take to reach a Level II condition (over 10 square feet)?

Solutions

1. The half-life of a radioactive substance is 10 days and there are 5 g present initially. Find an expression for the amount of material left after t days have elapsed. Show the process you used clearly, and do not begin with a formula. The idea is for you to end with a formula, not begin with it.

Then, find an expression for the time it would take to have m amount of material left. Explain using mathematics and the expression you find for the time that it must be true that $t > 0$.

We shall let $m(t)$ be the amount of radioactive material (in mg) present after time t in days. The initial amount is $m(0) = m_0 = 5$ mg.

We can write down how much is present at specific times knowing the half-life, and then generalize to get a formula for the amount present at any time.

$$\begin{aligned}
m(0) &= m_0 = 5 \\
m(10) &= \frac{1}{2}m(0) = \frac{1}{2}m_0 \\
m(20) &= \frac{1}{2}m(10) = \frac{1}{2^2}m_0 \\
m(30) &= \frac{1}{2}m(20) = \frac{1}{2^3}m_0 \\
m(40) &= \frac{1}{2}m(30) = \frac{1}{2^4}m_0 \\
&\vdots \quad \text{generalize from the pattern} \\
m(t) &= \frac{1}{2^{t/10}}m_0 = m_0 \cdot 2^{-t/10} = 5 \cdot 2^{-t/10}
\end{aligned}$$

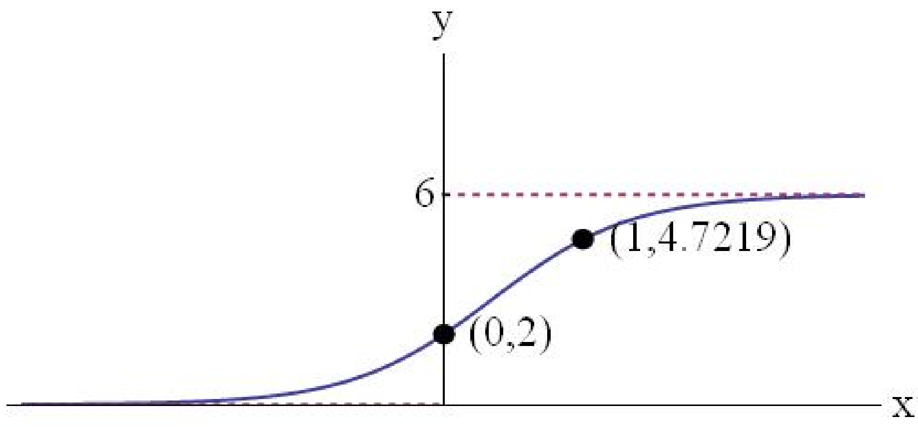
To get a formula for the time it takes to get to m, we should solve $m = 5 \cdot 2^{-t/10}$ for t.

$$\begin{aligned}
m &= 5 \cdot 2^{-t/10} \\
m/5 &= 2^{-t/10} \\
\ln(m/5) &= \ln(2^{-t/10}) \\
\ln(m/5) &= -\frac{t}{10} \cdot \ln(2) \\
-10\frac{\ln(m/5)}{\ln 2} &= t \\
t &= -10\frac{\ln(m/5)}{\ln 2}
\end{aligned}$$

It looks like $t < 0$, but since this is a radioactive decay problem we know the amount of material is getting smaller, so $m \leq 5$ (if $m = 5$, $t = -10\frac{\ln(5/5)}{\ln 2} = -10\frac{\ln 1}{\ln 2} = 0$).

If $m < 5$, then $\ln(m/5)$ will be the logarithm of a number less than one, which is negative, and combined with the -10 the overall result will be positive.

2. Determine an algebraic formula of the form $f(x) = \dfrac{c}{1 + ae^{-kx}}$ for the logistic function whose graph is shown below.



We can see that the logistic function has an asymptote at $y = 6$. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{c}{1 + ae^{-kx}} = \frac{c}{1+0} = c,$$

we must have $c = 6$.

Using the point $(0, 2)$, we can determine:

$$\begin{aligned} f(0) = \frac{6}{1 + ae^{-k(0)}} = \frac{6}{1+a} &= 2 \\ \frac{6}{1+a} &= 2 \\ \frac{1+a}{6} &= \frac{1}{2} \\ 1+a &= 3 \\ a &= 3 - 1 = 2 \end{aligned}$$

Finally, we can use the third point to determine k:

$$\begin{aligned} f(1) = \frac{6}{1 + 2e^{-k}} &= 4.7219 \\ \frac{1 + 2e^{-k}}{6} &= \frac{1}{4.7219} \\ 1 + 2e^{-k} &= \frac{6}{4.7219} \\ 2e^{-k} &= \frac{6}{4.7219} - 1 \\ e^{-k} &= \frac{3}{4.7219} - \frac{1}{2} = 0.135337 \end{aligned}$$

To go further algebraically, we need logarithms. Here is the algebraic solution using logarithms.

$$\begin{aligned}\ln e^{-k} &= \ln\left(\frac{3}{4.7219}-\frac{1}{2}\right)\\ -k &= \ln\left(\frac{3}{4.7219}-\frac{1}{2}\right)\\ k &= -\ln\left(\frac{3}{4.7219}-\frac{1}{2}\right) = 1.99998\end{aligned}$$

The logistic equation is

$$f(x) = \frac{6}{1+2e^{-1.99998x}}.$$

3. Consider depositing an amount of money $\$P$ into a savings account. After each month, the account earns $\frac{r}{12}\%$ interest based on the amount of money currently in the account. Therefore, r is the interest rate per year. This is called *compound interest* (you are earning interest on interest). Derive a formula for the amount of money in the account after one year.

Modify your formula for the case when interest is paid every day (assume 365 days in a year).

Modify your formula for the case when interest is paid every hour (assume 365 days in a year).

What would you guess the formula would be if interest was paid continuously?

The best way to answer this is from a table, where we look for a pattern. Let A be the amount of money we have in the account at time t in months:

$$\begin{aligned}A(0) &= P\\ A(1) &= P\left(1+\frac{r}{12}\right)^1\\ A(2) &= P\left(1+\frac{r}{12}\right)^1\left(1+\frac{r}{12}\right)^1 = P\left(1+\frac{r}{12}\right)^2\\ A(3) &= P\left(1+\frac{r}{12}\right)^2\left(1+\frac{r}{12}\right)^1 = P\left(1+\frac{r}{12}\right)^3\\ A(4) &= P\left(1+\frac{r}{12}\right)^3\left(1+\frac{r}{12}\right)^1 = P\left(1+\frac{r}{12}\right)^4\\ A(5) &= P\left(1+\frac{r}{12}\right)^4\left(1+\frac{r}{12}\right)^1 = P\left(1+\frac{r}{12}\right)^5\\ &\vdots\\ A(12) &= P\left(1+\frac{r}{12}\right)^{12}\end{aligned}$$

Therefore, if interest is earned every month, after one year we have:

$$A = P\left(1+\frac{r}{12}\right)^{12}.$$

If interest is earned every day, after one year we have:

$$A = P\left(1 + \frac{r}{365}\right)^{365}.$$

If interest is earned every hour, after one year we have:

$$A = P\left(1 + \frac{r}{8760}\right)^{8760}.$$

If interest is earned continuously, after one year we have:

$$A = P \lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{m}.$$

The fact that $\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{m} = e^r$ is one of the more interesting facts you will learn in calculus.

4. A researcher wishes to determine an exponential growth model for mold. She measures the amount of mold at $t = 0$ as 2mm^2. Two days later she finds 4mm^2.

Determine the exponential growth model $P(t) = a \cdot e^{kt}$. How long will the mold take to reach a Level II condition (over 10 square feet of mold)?

Use the two data points to determine the values of a and k in the model (two parameters, so we need two data points). Let's use units t days.

$$\begin{aligned}
P(0) = 2 &= a \cdot e^{k(0)} \\
2 &= a \cdot 1 \Rightarrow a = 2 \\
P(2) = 4 &= 2 \cdot e^{k(2)} \\
4 &= 2 \cdot e^{2k} \\
2 &= e^{2k} \\
\ln 2 &= \ln e^{2k} \\
\ln 2 &= 2k \Rightarrow k = \frac{1}{2}\ln 2
\end{aligned}$$

The exponential growth model is $P(t) = 2e^{\frac{t}{2}\ln 2}$.

We need to convert units to answer the second part. $10\text{ft}^2 = 929030\text{mm}^2$. Solve for t:

$$\begin{aligned}
P(t) = 929030 &= 2e^{\frac{t}{2}\ln 2} \\
464515 &= e^{\frac{t}{2}\ln 2} \\
\ln 464515 &= \ln e^{\frac{t}{2}\ln 2} \\
\ln 464515 &= \frac{t}{2}\ln 2 \Rightarrow t = 2\frac{\ln 464515}{\ln 2} \sim 38
\end{aligned}$$

In 38 days, or just over a month, you will be facing a Level II mold situation. Of course, the growing conditions may not be as ideal as during the initial growth, so you may actually have a bit longer. Regardless, any amount of mold is cause for concern.