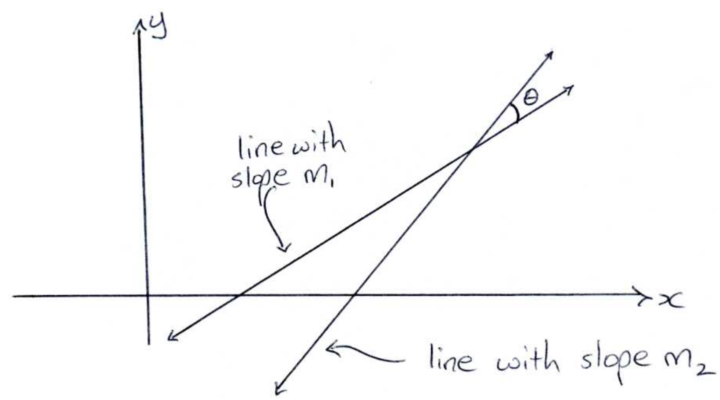
Questions

1. Convert the rectangular equation $(x+3)^2 + (y+3)^2 = 18$ into a polar equation, then solve for r.

2. The locations of two ships measured from a lighthouse are given in polar coordinates as $(3 \text{ miles}, 170°)$ and $(5 \text{ miles}, 150°)$. Find the distance between the two ships.

3. Show the angle θ between two lines with slopes m_1 and m_2 is given by the equation

$$\tan\theta = \frac{m_2 - m_1}{1 - m_2 m_1}$$



Hint: Figure out an equation that relates the slope of each line with the tangent of the angle it makes with the x-axis. Also try to figure out a relationship between all these angles, and maybe a trig identity for the tangent of a difference can get you to the final equation. This is a somewhat tricky, but very interesting problem!

Solutions

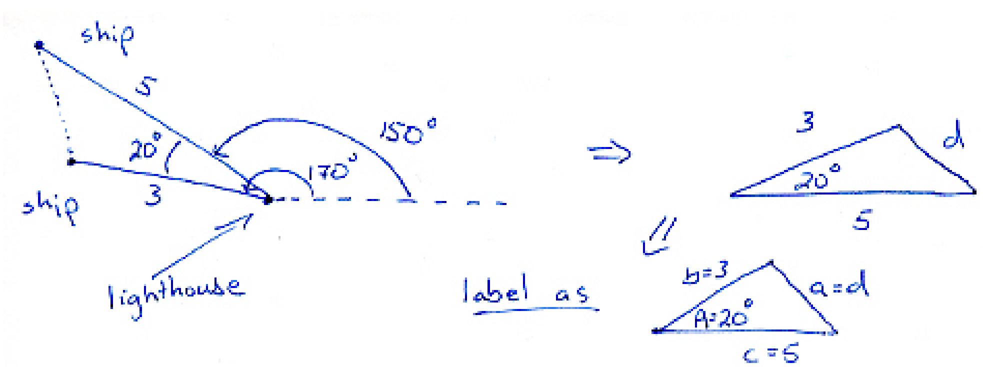
1. Convert the rectangular equation $(x+3)^2 + (y+3)^2 = 18$ into a polar equation, then solve for r.

We can do this if we make the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
(x+3)^2 + (y+3)^2 &= 18 \\
(r\cos\theta + 3)^2 + (r\sin\theta + 3)^2 &= 18 \\
(r^2\cos^2\theta + 9 + 6r\cos\theta) + (r^2\sin^2\theta + 9 + 6r\sin\theta) &= 18 \\
r^2(\cos^2\theta + \sin^2\theta) + 6r\cos\theta + 6r\sin\theta + 18 &= 18 \\
r^2 + 6r\cos\theta + 6r\sin\theta &= 0 \\
r + 6\cos\theta + 6\sin\theta &= 0 \\
r &= -6\cos\theta - 6\sin\theta
\end{aligned}$$

2. The locations of two ships measured from a lighthouse are given in polar coordinates as $(3 \text{ miles}, 170°)$ and $(5 \text{ miles}, 150°)$. Find the distance between the two ships.

Distance in polar coordinates comes from using the law of cosines. In our situation, we have a triangle that looks like the following to locate the position of the two ships.



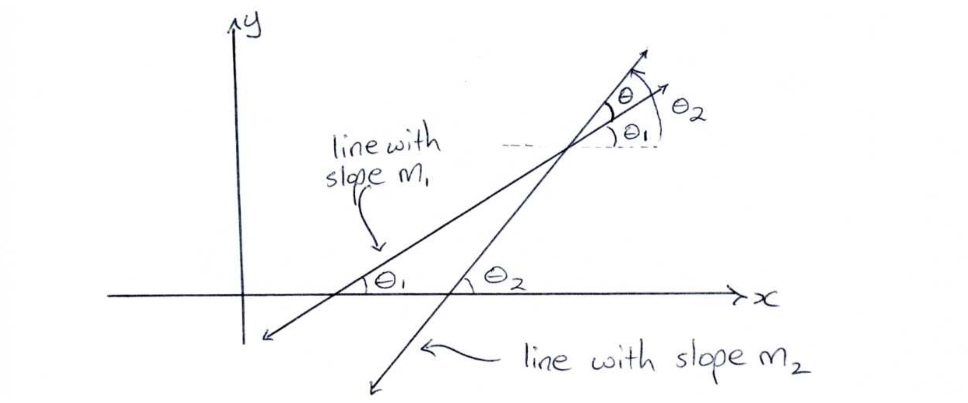
We can use the law of cosines to solve for d, this distance between the two ships.

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
d^2 &= 3^2 + 5^2 - 2(3)(5)\cos 20^{O} \\
&= 9 + 25 - 30(0.939693) \\
&= 5.80922 \\
d &= \sqrt{5.80922} = 2.41023
\end{aligned}$$

The distance between the ships is 2.41023 miles.

3. Show the angle θ between two lines with slopes m_1 and m_2 is given by the equation

$$\tan\theta = \frac{m_2 - m_1}{1 - m_2 m_1}$$



I've added some more information to the diagram, based on the hint to include the angle the lines make with the x-axis and to find a relationship between these three angles. This tells me

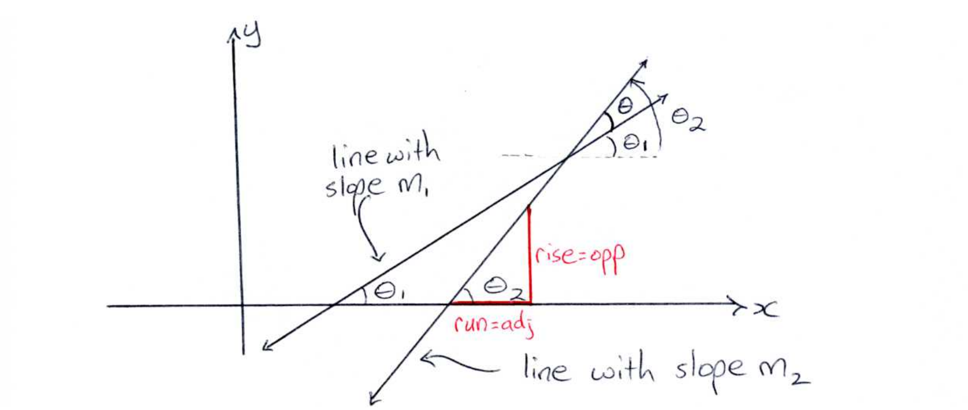
$$\theta_2 = \theta_1 + \theta \Rightarrow \theta = \theta_2 - \theta_1$$

OK, so another part of the hint says to use a tangent angle difference identity. There are a few to choose from, but the hint says to find one with the tangents of the angles θ_1 and θ_2, so let's get that.

$$\begin{aligned}\tan(\theta_2 - \theta_1) &= \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1}\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1}\end{aligned}$$

Multiply the numerator and denominator by $\dfrac{1}{\cos\theta_2\cos\theta_1}$ to get the second line.

This looks to have the correct form as the final answer we want! It appears that if $m_2 = \tan\theta_2$ and similarly for θ_1 and m_1, we would be done. Does this make sense? Another addition to the diagram shows it does.



Notice that for the red triangle I've drawn, it is true that

$$\tan\theta_2 = \frac{\text{opp}}{\text{adj}} = \frac{\text{rise}}{\text{run}} = m_2.$$

Similarly, $\tan\theta_1 = m_1$. So we have successfully shown that the angle θ between the two lines must satisfy

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1) \\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} \\ \tan\theta &= \frac{m_2 - m_1}{1 + m_2 m_1}\end{aligned}$$

That was fun!