Questions

1. Prove the identity $\dfrac{\tan x}{\sec x - 1} = \dfrac{\sec x + 1}{\tan x}$.

2. Let θ be any number that is in the domain of all six trigonometric functions. Explain why the natural logarithms of all six basic trig functions of θ sum to zero.

3. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\sin^5 x = (1 - 2\cos^2 x + \cos^4 x)\sin x$$

4. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\frac{1}{\tan\gamma} + \tan\gamma = \sec\gamma\csc\gamma$$

5. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\ln(\sec y + \tan y) + \ln(\sec y - \tan y) = 0$$

Solutions

1. Prove the identity $\dfrac{\tan x}{\sec x - 1} = \dfrac{\sec x + 1}{\tan x}$.

$$\begin{aligned}
\frac{\tan x}{\sec x - 1} &= \left(\frac{\tan x}{\sec x - 1}\right) \cdot 1 \\
&= \left(\frac{\tan x}{\sec x - 1}\right) \cdot \frac{\sec x + 1}{\sec x + 1} \\
&= \frac{\tan x(\sec x + 1)}{(\sec x - 1)(\sec x + 1)} \\
&= \frac{\tan x(\sec x + 1)}{\sec^2 x - 1} \\
&= \frac{\tan x(\sec x + 1)}{\tan^2 x}, \qquad \text{using } 1 + \tan^2 x = \sec^2 x \\
&= \frac{\sec x + 1}{\tan x}
\end{aligned}$$

2. Let θ be any number that is in the domain of all six trigonometric functions. Explain why the natural logarithms of all six basic trig functions of θ sum to zero.

Consider the two trig functions sine and cosecant. Since the cosecant is related to the sine by the reciprocal, when we take the logarithm of these two trig functions of θ and add them, we will get the natural logarithm of 1 (using the logarithm rules, the sum of the logarithms is the logarithm of the product). The natural logarithm of 1 is zero.

We can do the same thing for the cosine and secant pair, and the tangent and cotangent pair.

Therefore, the sum of the natural logarithms of all six basic trig functions of θ sum to zero.

The following shows this using mathematics:

$$\begin{aligned}
\ln(\cos\theta) + \ln(\sec\theta) + \ln(\sin\theta) + \ln(\csc\theta) + \ln(\tan\theta) + \ln(\cot\theta) &= \ln(\cos\theta \sec\theta \sin\theta \csc\theta \tan\theta \cot\theta) \\
&= \ln\left(\cos\theta \frac{1}{\cos\theta} \sin\theta \frac{1}{\sin\theta} \tan\theta \frac{1}{\tan\theta}\right) \\
&= \ln(1) \\
&= 0
\end{aligned}$$

Note that in this example you could have difficulty if you tried to evaluate a part like $\ln(\cos\theta)$ at an angle θ where $\cos\theta < 0$. However, if you extend your set of numbers from all real numbers to all complex numbers, then the result still holds.

3. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\sin^5 x = (1 - 2\cos^2 x + \cos^4 x)\sin x$$

$$\begin{aligned}
\sin^5 x &= \sin x \sin^4 x \\
&= \sin x(\sin^2 x)^2 \\
&= \sin x(1 - \cos^2 x)^2 \\
&= \sin x(1 - 2\cos^2 x + \cos^4 x)
\end{aligned}$$

4. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\frac{1}{\tan\gamma} + \tan\gamma = \sec\gamma\csc\gamma$$

$$\begin{aligned}
\frac{1}{\tan\gamma} + \tan\gamma &= \frac{\cos\gamma}{\sin\gamma} + \frac{\sin\gamma}{\cos\gamma} \\
&= \frac{\cos^2\gamma}{\cos\gamma\sin\gamma} + \frac{\sin^2\gamma}{\cos\gamma\sin\gamma} \\
&= \frac{\cos^2\gamma + \sin^2\gamma}{\cos\gamma\sin\gamma} \\
&= \frac{1}{\cos\gamma\sin\gamma} \\
&= \frac{1}{\cos\gamma} \cdot \frac{1}{\sin\gamma} \\
&= \sec\gamma \cdot \csc\gamma
\end{aligned}$$

5. Prove the algebraic identity by starting with the left hand side of the expression and supplying a sequence of equivalent expressions that ends with the right hand side of the expression.

$$\ln(\sec y + \tan y) + \ln(\sec y - \tan y) = 0$$

$$\begin{aligned}
\ln(\sec y + \tan y) + \ln(\sec y - \tan y) &= \ln\Big[(\sec y + \tan y)(\sec y - \tan y)\Big] \\
&= \ln\Big[\sec^2 y - \tan^2 y\Big] \\
&= \ln\Big[1\Big] \\
&= 0
\end{aligned}$$