Sketch Each of the following functions:

$$f(x) = \frac{(2x-1)^3(7-x)}{x(10-3x)^2}$$

$$f(x) = \frac{9x^2-12x+4}{16-x^2}$$

$$f(x) = \frac{(4x^2-1)^3}{x+1}$$

$$f(x) = \frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{x}$$

$$f(x) = \frac{8x^3+1}{2x+1}$$

$$f(x) = \frac{(3x^3-2x^2-3x+2)(x-1)}{x+4}$$

Ex sketch $f(x) = \dfrac{(2x-1)^3(7-x)}{x(10-3x)^2}$

f is already factored.

End behaviour: If $|x|$ is large, $f(x) \sim \dfrac{(2x)^3(-x)}{x(-3x)^2} = \dfrac{-8x^4}{9x^3} = -\dfrac{8}{9}x$.

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$

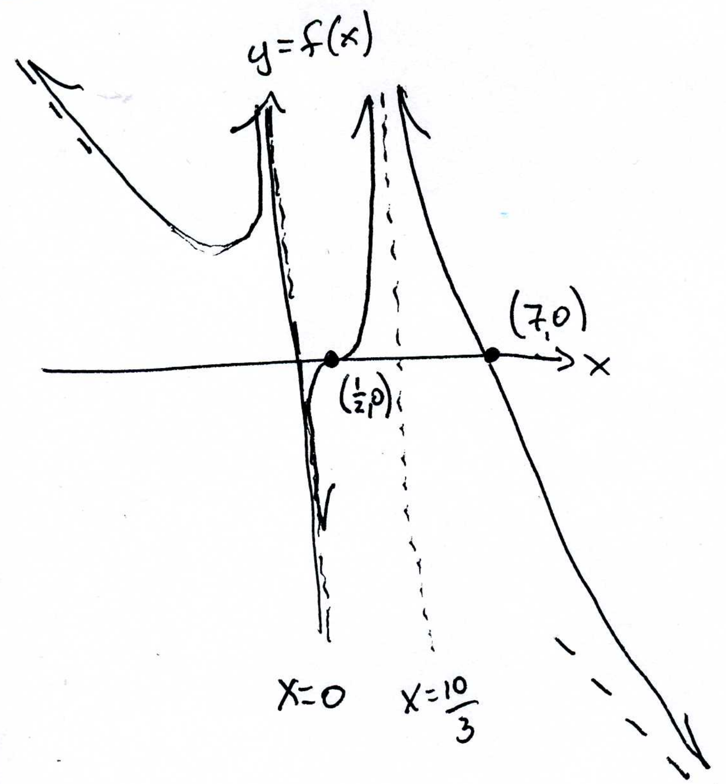
since there is a slant asymptote with slope $-8/9$.

zeros: $2x-1=0 \to x=\frac{1}{2}$ multiplicity 3 (odd) so f changes sign. Since multiplicity >1, f will be horizontal at $x=\frac{1}{2}$.

$7-x=0 \to x=7$ multiplicity 1 (odd), so f changes sign.

Vertical asymptotes: $x=0$ multiplicity 1 (odd) so f changes sign.

$10-3x=0 \to x=10/3$ multiplicity 2 (even) so f does not change sign.



slant asymptote with slope $-8/9$.

From the sketch, we can see that

$\lim_{x \to 10/3^+} f(x) = \infty$

$\lim_{x \to \frac{10}{3}^-} f(x) = \infty$

$\lim_{x \to 0^+} f(x) = -\infty$

$\lim_{x \to 0^-} f(x) = \infty$

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$

Domain $x \in (-\infty,0) \cup (0,\frac{10}{3}) \cup (\frac{10}{3},\infty)$

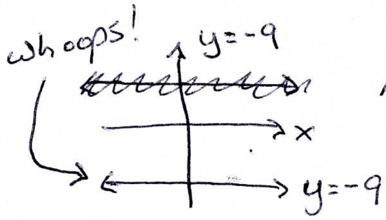
Ex Sketch $f(x) = \dfrac{9x^2-12x+4}{16-x^2}$

Factor $9x^2-12x+4 = (3x-2)^2$ perfect square

$16-x^2 = (4-x)(4+x)$ difference of squares

$$f(x) = \frac{(3x-2)^2}{(4-x)(4+x)}$$

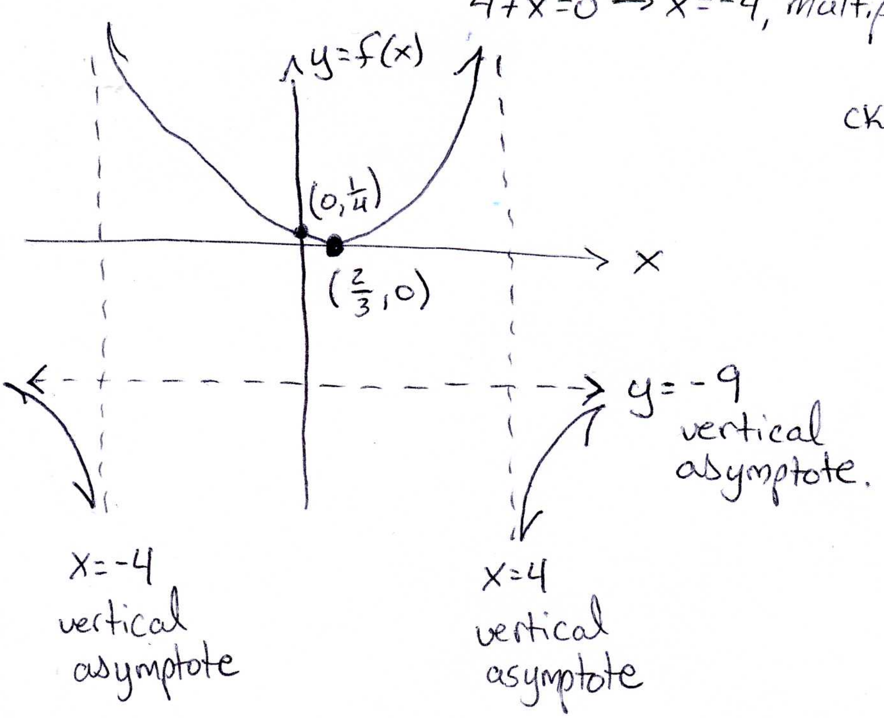
End Behaviour: If $|x|$ is large, $f(x) \sim \dfrac{9x^2}{-x^2} = -9$. Therefore there is a horizontal asymptote of $y=-9$.



Note: $\lim\limits_{x \to \pm\infty} f(x) = -9$

Zeros: $3x-2=0 \rightarrow x=\frac{2}{3}$, multiplicity 2 (even) so f does not change sign.

Vertical asymptotes: $4-x=0 \rightarrow x=4$, multiplicity 1 (odd) so f changes sign.
$4+x=0 \rightarrow x=-4$, multiplicity 1 (odd) so f changes sign.



Check: $f(0) = \frac{4}{16} = \frac{1}{4}$

so y intercept is 1/4, which agrees with our sketch.

From the sketch we see that

$\lim\limits_{x \to 4^+} f(x) = -\infty$

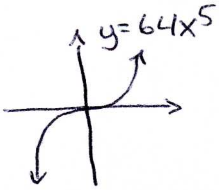
$\lim\limits_{x \to 4^-} f(x) = \infty$

$\lim\limits_{x \to -4^+} f(x) = \infty$

$\lim\limits_{x \to -4^-} f(x) = -\infty$

Ex) Sketch $f(x) = \dfrac{(4x^2-1)^3}{x+1}$

End behaviour: If $|x|$ is large, $f(x) \sim \dfrac{(4x^2)^3}{x} = \dfrac{64x^6}{x} = 64x^5$.



Note: $\lim_{x\to\infty} f(x) = \infty$ $\quad \lim_{x\to-\infty} f(x) = -\infty$. No horizontal or slant asymptotes.

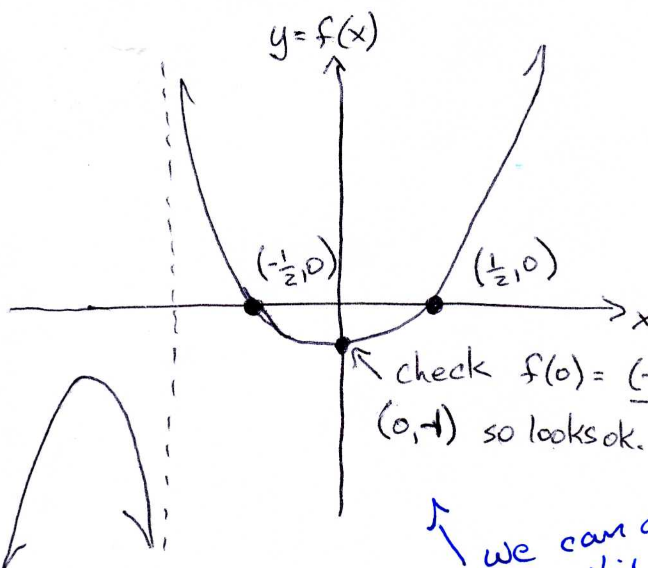
we need to factor before we find zeros.

$$4x^2-1 = (2x)^2 - 1^2 = (2x-1)(2x+1) \quad \text{difference of squares.}$$

$$f(x) = \frac{(2x-1)^3(2x+1)^3}{x+1}$$

Zeros: $2x-1=0 \to x=\frac{1}{2}$, multiplicity 3 (odd) so f changes sign.
$2x+1=0 \to x=-\frac{1}{2}$, multiplicity 3 (odd) so f changes sign.

vertical asymptotes: $x+1=0 \to x=-1$, multiplicity 1 (odd) so f changes sign.



check $f(0) = \frac{(-1)^3}{1} = -1$
(0,-1) so looks ok.

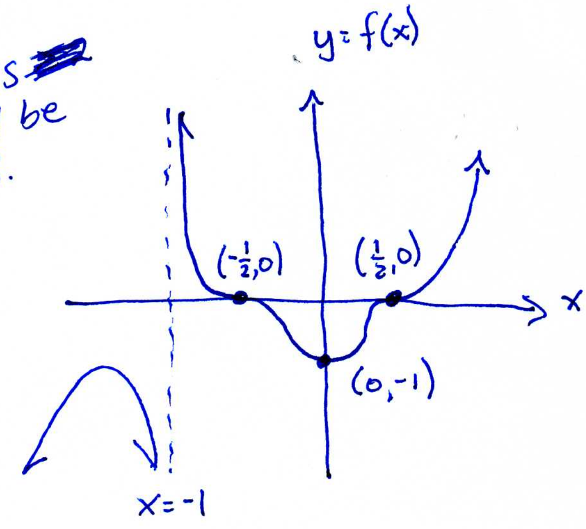
x=-1
vertical asymptote

From the sketch we can see that:

$$\lim_{x\to-1^+} f(x) = \infty$$

$$\lim_{x\to-1^-} f(x) = -\infty$$

we can do better than this! Since multiplicity of zeros is $3 \geq 2$, the function will be horizontal at the zeros.



Ex sketch $f(x) = \frac{1}{3x-1} + \frac{1}{(3x-1)^2} + \frac{1}{x}$

First get a common denominator:

$$f(x) = \frac{1}{3x-1} \cdot \frac{(3x-1)x}{(3x-1)x} + \frac{1}{(3x-1)^2} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{(3x-1)^2}{(3x-1)^2}$$

$$= \frac{(3x-1)x + x + (3x-1)^2}{(3x-1)^2 x}$$

$$= \frac{3x^2 - x + x + 9x^2 - 6x + 1}{(3x-1)^2 x}$$

$$= \frac{12x^2 - 6x + 1}{(3x-1)^2 x}$$

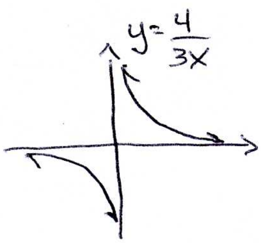
Factor $12x^2 - 6x + 1$: use quadratic formula:

$$x = \frac{6 \pm \sqrt{36 - 4(12)(1)}}{2 \cdot 12}$$

$$= \frac{6 \pm \sqrt{-12}}{24}$$

no real roots, so quadratic is irreducible.

End behaviour: If $|x|$ is large, $f(x) \sim \frac{12x^2}{(3x)^2 x} = \frac{12x^2}{9x^3} = \frac{4}{3x}$



Note: $\lim_{x \to \infty} f(x) = 0$

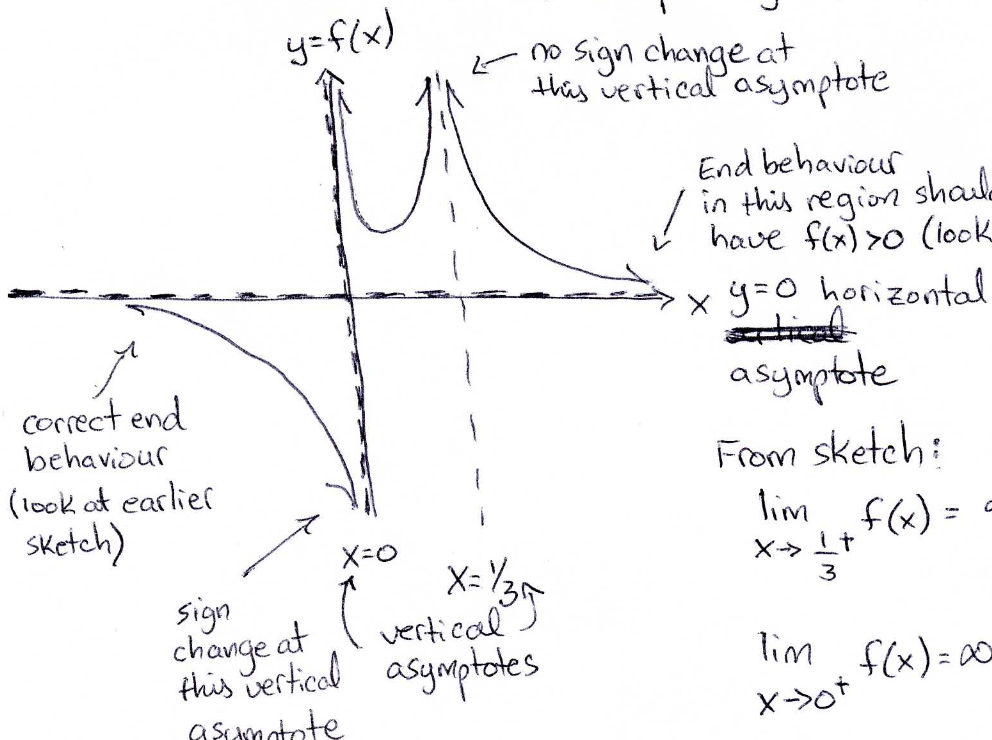
$\lim_{x \to -\infty} f(x) = 0$

So f has a horizontal asymptote of $y = 0$.

Zeros: no zeros since $12x^2 - 6x + 1$ is irreducible.

vertical asymptotes: $(3x-1) = 0 \rightarrow x = \frac{1}{3}$, multiplicity 2 (even) so f does not change sign.

$x = 0$, multiplicity 1 (odd) so f changes sign.



From sketch:

$\lim_{x \to \frac{1}{3}^+} f(x) = \infty$ $\quad \lim_{x \to \frac{1}{3}^-} f(x) = \infty$

since they both approach same limit, we can combine these and say $\lim_{x \to \frac{1}{3}} f(x) = \infty$

$\lim_{x \to 0^+} f(x) = \infty$ $\quad \lim_{x \to 0^-} f(x) = -\infty$

Ex Sketch $f(x) = \dfrac{8x^3+1}{2x+1}$

Factor $8x^3+1$. Sum of cubes. use formula, or notice $8x^3+1\Big|_{x=-1/2} = 0 \Rightarrow x+\frac{1}{2}$ is a factor.

$$
\begin{array}{r l}
 & 8x^2 - 4x + 2 \\
x+\frac{1}{2} & \overline{\big)\, 8x^3 + 0x^2 + 0x + 1} \\
 & 8x^3 + 4x^2 \quad \text{subtract} \\
 & \overline{-4x^2 + 0x} \\
 & -4x^2 - 2x \\
 & \overline{2x + 1} \\
 & 2x + 1 \\
 & \overline{\qquad 0}
\end{array}
$$

Note: $8x^2-4x+2$ is irreducible since

$$x = \frac{4 \pm \sqrt{16-4(8)(2)}}{16} = \frac{4 \pm \sqrt{-48}}{16} \text{ not a real number.}$$

$$f(x) = \frac{(x+\frac{1}{2})(8x^2-4x+2)}{2x+1}$$

$$= \frac{(x+\frac{1}{2})\,2\,(4x^2-2x+1)}{2x+1}$$

$$= \frac{\cancel{(2x+1)}(4x^2-2x+1)}{\cancel{(2x+1)}}$$

Note: numerator and denominator are both zero when $x=-1/2$, which means we have a hole at $x=-1/2$. Rewrite to cancel the factor.

$= 4x^2-2x+1$, $\quad 2x+1 \neq 0 \Rightarrow x \neq -1/2$. This is a quadratic

- opens up since $a=4>0$.
- no zeros since $4x^2-2x+1$ is irreducible
- get vertex by completing the square.
- hole at $x=-1/2$

$$y = f(-\tfrac{1}{2}) = 4(-\tfrac{1}{2})^2 - 2(-\tfrac{1}{2}) + 1 = 3$$

hole at $(-\frac{1}{2}, 3)$.

$$4x^2-2x+1 = 4\left(x^2 - \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2\right) + 1$$

$$= 4\left([x-\tfrac{1}{4}]^2 - \tfrac{1}{16}\right) + 1$$

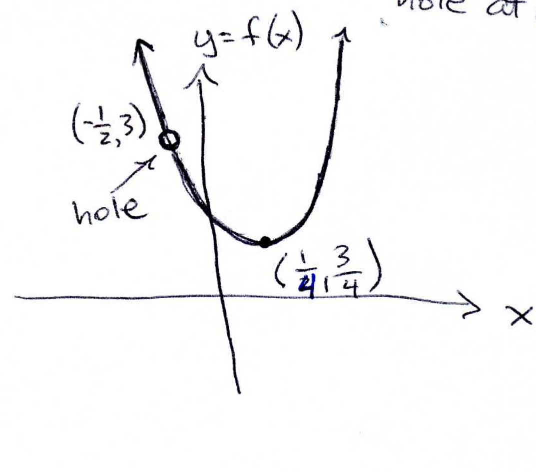
$$= 4[x-\tfrac{1}{4}]^2 - \tfrac{4}{16} + 1$$

$$= 4[x-\tfrac{1}{4}]^2 + \tfrac{3}{4}$$

So $f(x) = 4[x-\frac{1}{4}]^2 + \frac{3}{4}$, $x \neq -1/2$

vertex form $y = a(x-h)^2+k$.

$\Rightarrow (h,k) = (\frac{1}{4}, \frac{3}{4})$



$\boxed{\text{Ex}}$ Sketch $f(x) = \dfrac{(3x^3-2x^2-3x+2)(x-1)}{x+4}$

Factor $3x^3-2x^2-3x+2$

Factors of $a_0 = +2$: $\pm1, \pm2$
Factors of $a_3 = 3$: $\pm1, \pm3$

Possible rational factors: $\pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}$.

$3(1)^3-2(1)^2-3(1)-2=0$, so $x-1$ will factor.

$$\begin{array}{r|l} & 3x^2+x-2 \\ \hline x-1 & 3x^3-2x^2-3x+2 \\ & 3x^3-3x^2 \\ \hline & x^2-3x \\ & x^2-x \\ \hline & -2x+2 \\ & -2x+2 \\ \hline & 0 \end{array}$$

subtract

so $3x^3-2x^2-3x+2 = (x-1)(3x^2+x-2)$
$= (x-1)(3x-2)(x+1)$

If you need to factor $3x^2+x-2$ using quadratic formula, here are the details:

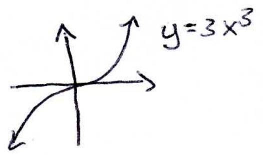
$$x = \frac{-1 \pm \sqrt{1-4(3)(-2)}}{6} = \frac{-1\pm5}{6} = -1 \text{ or } \frac{2}{3}$$

factor $(x+1)$ factor $x-\frac{2}{3}$

So $3x^2+x-2 = 3(x+1)(x-\frac{2}{3})$
$= (x+1)(3x-2)$.

So $f(x) = \dfrac{(x-1)(3x-2)(x+1)(x-1)}{x+4}$
$= \dfrac{(3x-2)(x+1)(x-1)^2}{x+4}$ Collect $(x-1)$ factors together.

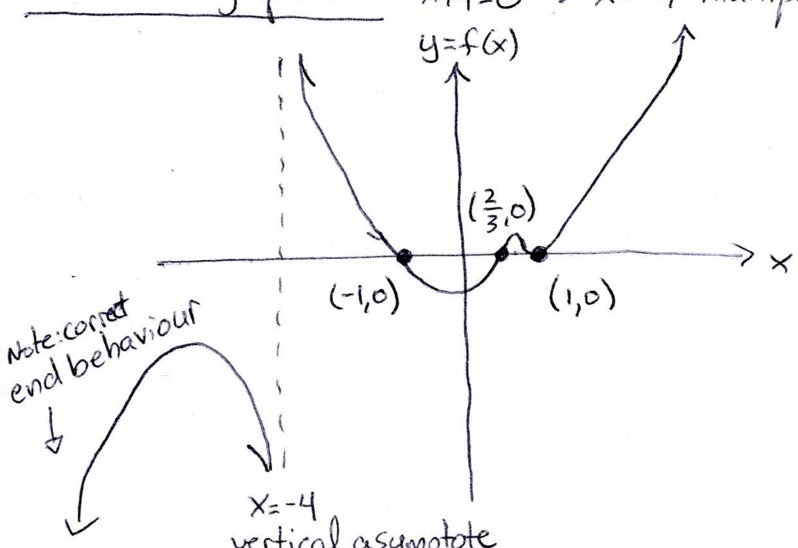
End behaviour: If $|x|$ is large, $f(x) \sim \dfrac{(3x)(x)(x)^2}{x} = 3x^3$.



so $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Zeros: $3x-2=0 \to x=2/3$ } multiplicity 1 (odd) so f changes sign.
$x+1=0 \to x=-1$
$x-1=0 \to x=1$ multiplicity 2 (even) so f does not change sign.

vertical asymptotes: $x+4=0 \to x=-4$ multiplicity 1 (odd) so f changes sign.



From the sketch we can see that

$$\lim_{x\to-4^+} f(x) = \infty$$

$$\lim_{x\to-4^-} f(x) = -\infty$$