Questions

1. Find the value of $\sin\left(-\frac{\pi}{12}\right)$ exactly.

2. Prove the identity $\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta$.

3. Prove the identity $\sin 4x+\sin 2x=2\sin 3x\cos x$.

4. Find the value of $\sin\left(-\frac{5\pi}{12}\right)$ exactly by using the sine of a sum identity. This problem shows you a method to determine exactly the trig functions at angles other than the special angles on the unit circle.

Solutions

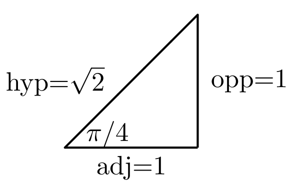
1. Find the value of $\sin\left(-\frac{\pi}{12}\right)$ exactly.

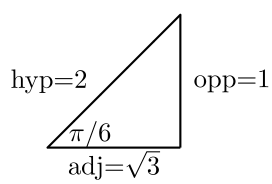
First, we need to figure out how to relate $-\pi/12$ to some of our special angles, since we are told to find this answer exactly.

$$\frac{-\pi}{12} = \frac{-2\pi}{24} = \frac{4\pi - 6\pi}{24} = \frac{\pi}{6} - \frac{\pi}{4}.$$

Therefore,

$$\begin{aligned}
\sin\left(-\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right) \\
&= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right), \quad \text{use } \sin(u-v) = \sin u \cos v - \cos u \sin v \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right), \quad \text{using reference triangles below} \\
&= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} \\
&= \frac{1-\sqrt{3}}{2\sqrt{2}}
\end{aligned}$$





2. Prove the identity $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$.

$$\begin{aligned}
\cos\left(\theta + \frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right)\cos\theta - \sin\left(\frac{\pi}{2}\right)\sin\theta, \quad \text{using } \cos(u+v) = \cos u \cos v - \sin u \sin v \\
&= (0)\cos\theta - (1)\sin\theta, \quad \text{using the unit circle to evaluate the trig functions of } pi/2. \\
&= -\sin\theta
\end{aligned}$$

3. Prove the identity $\sin 4x + \sin 2x = 2\sin 3x \cos x$.

$$
\begin{aligned}
\sin 4x + \sin 2x &= \sin(3x + x) + \sin(x + x), \\
&= (\sin 3x \cos x + \cos 3x \sin x) + (\sin x \cos x + \cos x \sin x), \quad \text{use } \sin(u+v) = \sin u \cos v + \cos u \sin v \text{ twice.} \\
&= \sin 3x \cos x + (\cos 3x) \sin x + 2\sin x \cos x, \\
&= \sin 3x \cos x + (\cos(2x + x)) \sin x + 2\sin x \cos x \\
&= \sin 3x \cos x + (\cos 2x \cos x - \sin 2x \sin x) \sin x + 2\sin x \cos x, \quad \text{using } \cos(u+v) = \cos u \cos v - \sin u \sin v \\
&= \sin 3x \cos x + \cos 2x \cos x \sin x - \sin 2x \sin^2 x + 2\sin x \cos x \\
&= \sin 3x \cos x + (\cos 2x) \cos x \sin x - (\sin 2x) \sin^2 x + 2\sin x \cos x \\
&= \sin 3x \cos x + (\cos(x + x)) \cos x \sin x - (\sin(x + x)) \sin^2 x + 2\sin x \cos x \\
&= \sin 3x \cos x + (\cos x \cos x - \sin x \sin x) \cos x \sin x - (\sin x \cos x + \cos x \sin x) \sin^2 x + 2\sin x \cos x \\
&= \sin 3x \cos x + \cos^3 x \sin x - 3\sin^3 x \cos x + 2\sin x \cos x
\end{aligned}
$$

We can see that this is very nasty looking. Let's start over.

$$
\begin{aligned}
\sin 4x + \sin 2x &= \sin(3x + x) + \sin(3x - x), \\
&= (\sin 3x \cos x + \cos 3x \sin x) + (\sin 3x \cos x - \cos 3x \sin x), \quad \text{use } \sin(u \pm v) = \sin u \cos v \pm \cos u \sin v \\
&= 2\sin 3x \cos x
\end{aligned}
$$

That was certainly easier! In this case, we exploited some of the hidden symmetry in the problem by writing $2x = 3x - 1$ rather than $2x = x + x$.

4. Find the value of $\sin\left(-\frac{5\pi}{12}\right)$ exactly by using the sine of a sum identity. This problem shows you a method to determine exactly the trig functions at angles other than the special angles on the unit circle.

First, we need to figure out how to relate $-5\pi/12$ to some of our special angles, since we are told to find this answer exactly. We are told to use a sum formula, so the sum of two of our special angles should produce $-5\pi/12$.

$$\frac{-5\pi}{12} = \frac{-10\pi}{24} = \frac{-6\pi - 4\pi}{24} = -\frac{\pi}{4} - \frac{\pi}{6}.$$

Therefore,

$$\begin{aligned}
\sin\left(-\frac{5\pi}{12}\right) &= \sin\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{6}\right) + \cos\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{6}\right), \quad \text{use } \sin(u+v) = \sin u \cos v + \cos u \sin v \\
&= \left(-\sin\left(\frac{\pi}{4}\right)\right)\left(\cos\left(\frac{\pi}{6}\right)\right) + \left(\cos\left(\frac{\pi}{4}\right)\right)\left(-\sin\left(\frac{\pi}{6}\right)\right), \quad \text{use } \sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta \\
&= \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right), \quad \text{using reference triangles below} \\
&= -\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
&= -\frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$