Questions

1. Find $\csc\theta$ and $\cot\theta$ if $\tan\theta = -\dfrac{4}{3}$ and $\sin\theta > 0$.

2. An airplane flying at an altitude of 8000 ft passes directly over a group of hikers who are at 7400 ft. If θ is the angle of elevation from the hikers to the aircraft, find the distance d from the group to the aircraft for the given angle.

(a) $\theta = 45°$ (b) $\theta = 90°$ (c) $\theta = 140°$

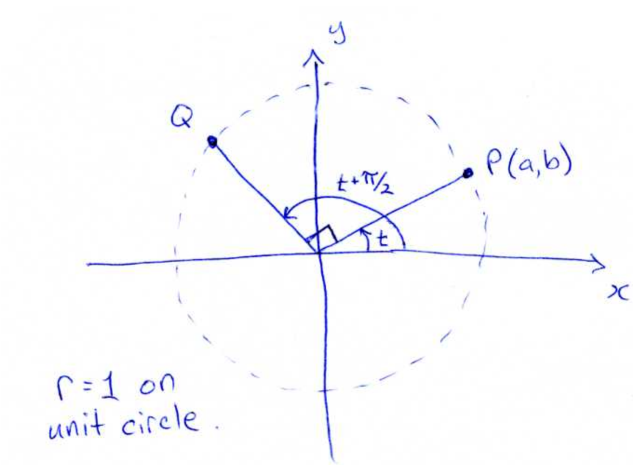
The angle of elevation is the angle through which the eye moves up from the horizontal to look at an object in the sky (if you have to look down, it is called the angle of depression).

3. Explain why $\sin(7\pi/6) = -1/2$.

4. Explain why $\cot(4\pi/3) = 1/\sqrt{3}$.

5. Using a unit circle, explain why $\sin(t + \pi/2) = \cos t$.

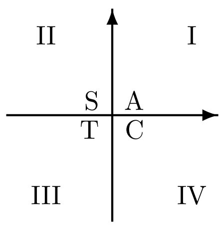
Hint: In the following diagram, you need to figure out the coordinates for Q (they should depend on a, b, the coordinates of P). Use the fact that the line through the origin and P is perpendicular to the line through the origin and Q to do this (and you'll need the equations of these lines).



Solutions

1. Find $\csc\theta$ and $\cot\theta$ if $\tan\theta = -\frac{4}{3}$ and $\sin\theta > 0$.

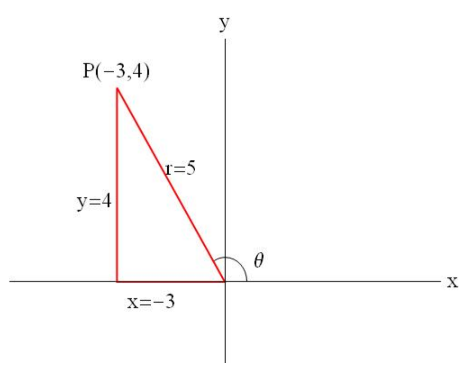
Let's solve this by drawing a coordinate system. First, we need to figure out what Quadrant P lies in.



When $\sin\theta = \frac{y}{r} > 0 \Rightarrow P$ is in either QI or QII.
When $\tan\theta = \frac{y}{x} < 0 \Rightarrow P$ is in either QII or QIV.

Therefore, we must be in Quadrant II (so both above facts are true). Our angle has a terminal side in the second quadrant.

Sketch, and use $\tan\theta = y/x = 4/(-3)$ to determine $x = -3$ and $y = 4$ (if $y = -4$ and $x = 3$ we would be in the wrong quadrant).



where we have found r using the Pythagorean theorem.

$$\begin{aligned} x &= -3 \\ y &= 4 \\ r &= \sqrt{x^2+y^2} = \sqrt{(-3)^2+(4)^2} = \sqrt{25} = 5 \end{aligned}$$

Therefore, using the definitions of the trig functions,

$$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4}$$

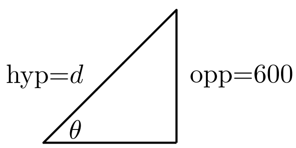
$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{5}{4}$$

2. An airplane flying at an altitude of 8000 ft passes directly over a group of hikers who are at 7400 ft. If θ is the angle of elevation from the hikers to the aircraft, find the distance d from the group to the aircraft for the given angle.

(a) $\theta = 45°$ (b) $\theta = 90°$ (c) $\theta = 140°$

The angle of elevation is the angle through which the eye moves up from the horizontal to look at an object in the sky (if you have to look down, it is called the angle of depression).

The aircraft is 600 ft above the hikers when it passes directly overhead. Assuming that the aircraft's altitude does not change, this means that we have a reference triangle for the situation given by:



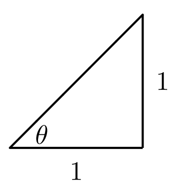
From the reference triangle, we have

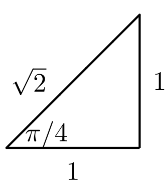
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{600}{d}.$$

Therefore, $d = \dfrac{600}{\sin\theta}$. Now we have to evaluate the sine of the angle. I am including all of the details here to remind myself of how we get the sine for special angles.

(a) $\theta = 45^{\text{o}} = \dfrac{\pi}{4}$ radians:

Consider the isosceles (a triangle with two equal sides) right triangle given below.





The angle here must be $\pi/4$ radians, since this triangle is half of a square of side length 1.

$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

Therefore, $d = \dfrac{600}{\sin(\pi/4)} = \dfrac{600}{\left(\frac{1}{\sqrt{2}}\right)} = 600\sqrt{2} \sim 848.528$ ft.

(b) $\theta = 90^{\text{o}} = \dfrac{\pi}{2}$ radians:

If we think of the unit circle, we realize that the sine of $\pi/2$ is 1, since a rotation of $\pi/2$ puts us at the point $(0, 1)$, and the sine is the y value of the points on the unit circle.

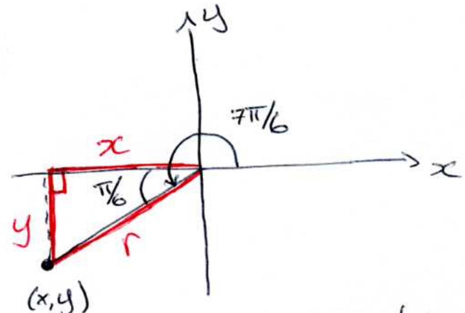
Therefore, $d = \dfrac{600}{\sin(\pi/2)} = \dfrac{600}{(1)} = 600$ ft. This is when the aircraft is directly overhead.

(c) $\theta = 140^{\text{o}} = \dfrac{7\pi}{9}$ radians:

This is one of the angles we have to use a calculator to figure out. Therefore, $d = \dfrac{600}{\sin(7\pi/9)} \sim 933.434$ ft.

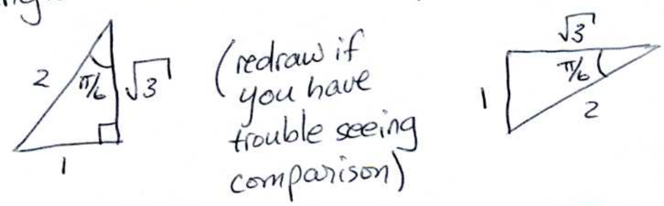
3. Explain why $\sin(7\pi/6) = -1/2$.

Sketch: $\dfrac{7\pi}{6} = \pi + \dfrac{\pi}{6}$



The red triangle is one of our special triangles:

Compare



so $r = 2$, $x = \sqrt{3}$, $y = 1$

<u>But</u> in Quadrant III, $x < 0$, $y < 0$

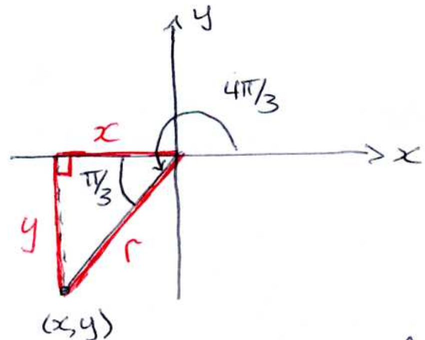
so use $r = 2$, $x = -\sqrt{3}$, $y = -1$.

From definition, $\sin\theta = \dfrac{y}{r}$

$\sin\left(7\pi/6\right) = -\dfrac{1}{2}$.

4. Explain why $\cot(4\pi/3) = 1/\sqrt{3}$.

sketch $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$



The red triangle is one of of our special triangles:

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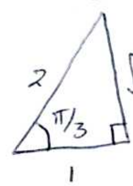

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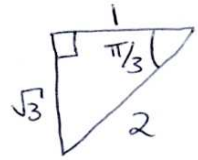


The angle puts us in Quadrant III, so $x < 0$, $y < 0$.

$$\Rightarrow \quad x = -1, \quad y = -\sqrt{3}, \quad r = 2$$

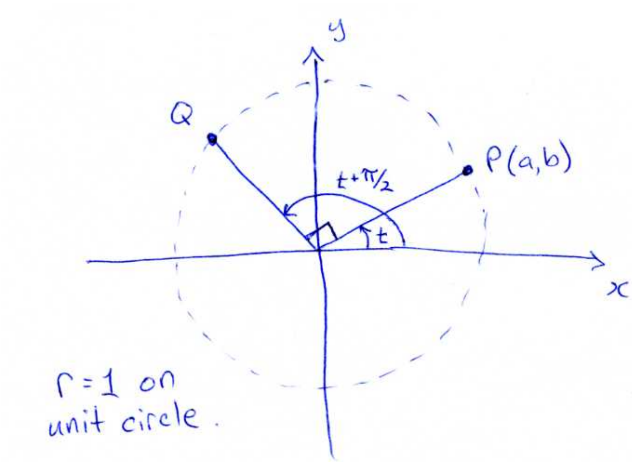
From definition:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{(y/x)} = \frac{x}{y}$$

$$\cot(4\pi/3) = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

5. Using a unit circle, explain why $\sin(t + \pi/2) = \cos t$.

Hint: In the following diagram, you need to figure out the coordinates for Q (they should depend on a, b, the coordinates of P). Use the fact that the line through the origin and P is perpendicular to the line through the origin and Q to do this (and you'll need the equations of these lines).



From P: $\cos t = \dfrac{x}{r} = \dfrac{a}{1} = a$.

Get the coordinates of Q:

Line through origin and P has slope b/a and y-intercept 0, so $y = \dfrac{b}{a}x$.

Line through origin and Q is perpendicular to this line, so it has slope $-a/b$ (negative reciprocal) and equation $y = -\dfrac{a}{b}x$.

Since slope is rise/run, we have rise$= a$ and run$=-b$ (minus sign goes with run based on diagram).

So Q has coordinates $Q(-b, a)$.

Aside: Only concern here is why Q doesn't have coordinates like $Q(-2b, 2a)$ (still has slope $-a/b$)? In this case, $r = \sqrt{x^2 + y^2} = 2\sqrt{b^2 + y^2} = 2 \neq 1$ (it must be 1 since it is on unit circle) so we have a contradiction.

From Q: $\sin(t + \pi/2) = \dfrac{y}{r} = \dfrac{a}{1} = a$.

So $a = \cos t = \sin(t + \pi/2)$.

WE RULE!!