Geometric Series

Consider the following series: $1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

We need to figure out a way to write this without the $\cdots$. Here's how:

$$\begin{aligned}
s &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} \\
\text{subtract } s \text{ times } x \text{ from } s\text{: } \underline{sx} &= \underline{\quad\ x + x^2 + x^3 + \cdots + x^{n-1} + x^n} \\
s - sx &= 1 - 0 - 0 - 0 - \cdots - 0 - x^n \\
s - sx &= 1 - x^n
\end{aligned}$$

Now solve for s:

$$s(1-x) = 1 - x^n \Rightarrow s = \frac{1-x^n}{1-x} = \frac{x^n - 1}{x - 1}.$$

Therefore, a geometric series has the following sum: $1 + x + x^2 + x^3 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$.

Example

$$\begin{aligned}
\sum_{n=0}^{10} \left(\frac{1}{2}\right)^n &= \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{10} \\
&= 1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{11-1} \\
&\text{geometric series with } x = 1/2 \text{ and } n = 11 \\
&= \frac{x^n - 1}{x - 1} \\
&= \frac{(1/2)^{11} - 1}{(1/2) - 1} \\
&= \frac{2047}{1024}
\end{aligned}$$

This formula is most useful when the place you stop the sum n is not a number but a parameter, as in the case of The Savings Formula.