Modeling Weight and Pulse Rate of Mammals

From *Precalculus: Graphical, Numericl, Algebraic* by Demana, Waits, Foley & Kennedy Section 2.2 Problem 55 and 3.4 Problem 65. Data comes from A.J. Clark, Comparative Physiology of the Heart, New York: Macmillan, 1927.

For mammals and other warm-blooded animals to stay warm requires quite a bit of energy. Temperature loss is related to surface area (which is related to body weight) and temperature gain is related to circulation (which is related to pulse rate). Scientists have concluded that the pulse rate *r* of mammals is a power function of their body weight *w*.

Here is some data for the weight and pulse rate of mammals:

Out[120]//TableForm=

	Weight w (kg)	Pulse Rate r (beats/min)
rat	0.2	420
guinea pig	0.3	300
rabbit	2	205
small dog	5	120
large dog	30	85
sheep	50	70
human	70	72

We can do a scatterplot of the data and try to determine the best fit function.

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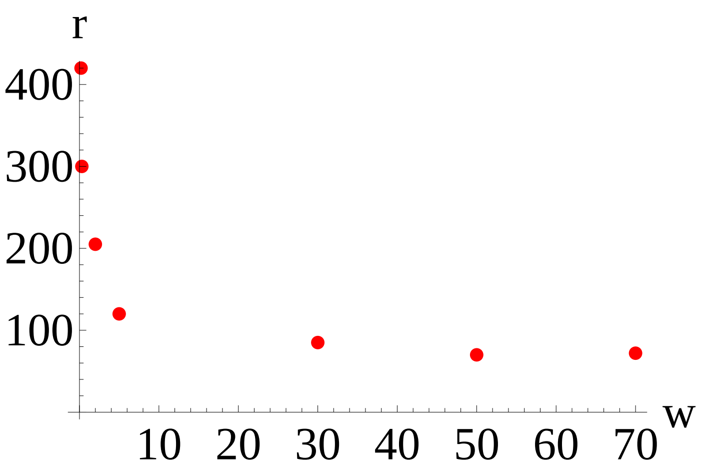

We can see that a power function might fit the data, let's try $r = f(w) = 300\, w^{-1/3}$. It has to be a negative power since the pulse decreases as the weight increases (an inverse variation). We picked 300 as the multiplier just to get something close to the data

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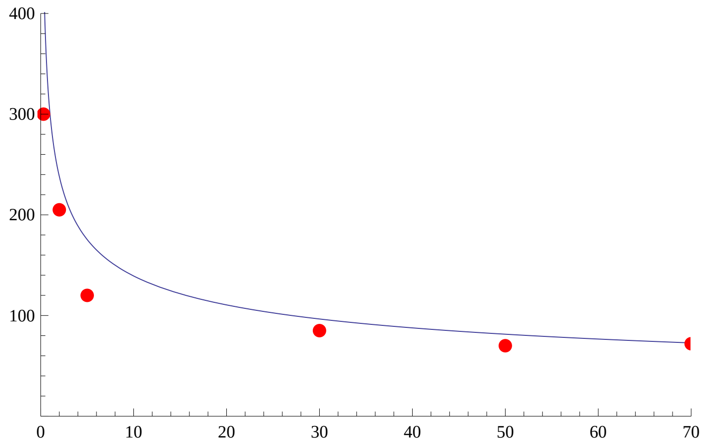

Hey, not bad. To get the best fit we would have to do some regression to fit a curve $r = a\,w^b$ to the data points, which would be difficult since we don't yet know how to do power regression. However, earlier in the semester we looked at how to do linear regression, which is used to fit $y = m\,x + b$ to data points.

The good news is we can use logarithms to turn $r = a\,w^b$ into a linear relationship!

$r = a\,w^b$ Take logarithm of both sides of the equation

$\ln(r) = \ln\left(a\,w^b\right)$ Use logarithm rule $\ln(x\,y) = \ln(x) + \ln(y)$

$\ln(r) = \ln\left(w^b\right) + \ln(a)$ Use logarithm rule $\ln(x^y) = y\ln(x)$

$\ln(r) = b\ln(w) + \ln(a)$

So, if we use data points $x = \ln(r)$ and $y = \ln(w)$ we can fit the line $y = m\,x + b$ and determine the relationship! The slope of the line wll be the power b, and the y-intercept is the logarithm of the proportinalility constant a. Here is the data with logarithms taken:

Out[125]//TableForm=

	x = ln(w)	y = ln(r)
rat	-1.60944	6.04025
guinea pig	-1.20397	5.70378
rabbit	0.693147	5.32301
small dog	1.60944	4.78749
large dog	3.4012	4.44265
sheep	3.91202	4.2485
human	4.2485	4.27667

Here is the scatterplot of the logarithmic data, which you can see looks nice and linear.

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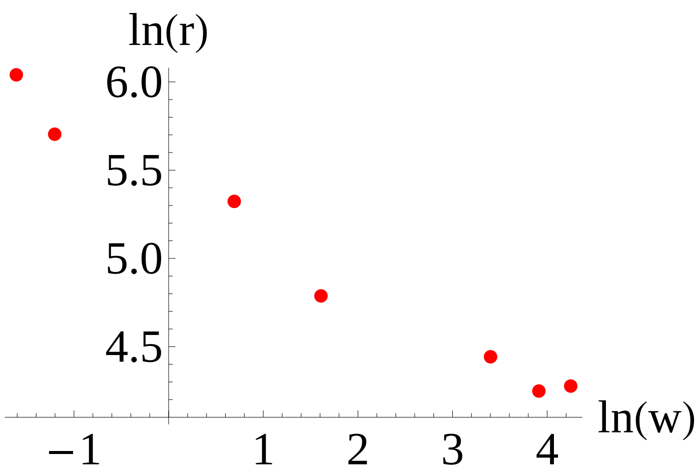

Using the equations we worked out earlier for fitting $y = m\,x + b$ to data points,

$$m = \frac{n \sum x\, y - \sum x \sum y}{n \sum x^2 - \left(\sum x\right)^2}$$

$$b = \frac{\sum y - m \sum x}{n}$$

We see that we are going to need to extend our data table a bit to be able to track $\sum x$, $\sum y$, $\sum x^2$, $\sum x\, y$.

Out[135]//TableForm=

	x=ln(w)	y=ln(r)	x^2	xy
rat	-1.60944	6.04025	2.59029	-9.72141
guinea pig	-1.20397	5.70378	1.44955	-6.8672
rabbit	0.693147	5.32301	0.480453	3.68963
small dog	1.60944	4.78749	2.59029	7.70517
large dog	3.4012	4.44265	11.5681	15.1103
sheep	3.91202	4.2485	15.3039	16.6202
human	4.2485	4.27667	18.0497	18.1694
TOTAL	11.0509	34.8224	52.0324	44.7061

Now, we can define the quantities we need:

$$m = \frac{n \sum x\, y - \sum x \sum y}{n \sum x^2 - \left(\sum x\right)^2} = \frac{(7)\,(44.7061) - (11.0509)\,(34.8224)}{(7)\,(52.0324) - (11.0509)^2} = -0.296881$$

$$b = \frac{\sum y - m \sum x}{n} = \frac{(34.8224) - (-0.296881)\,(11.0509)}{7} = 5.44331$$

$$y = -0.296881\, x + 5.44331$$

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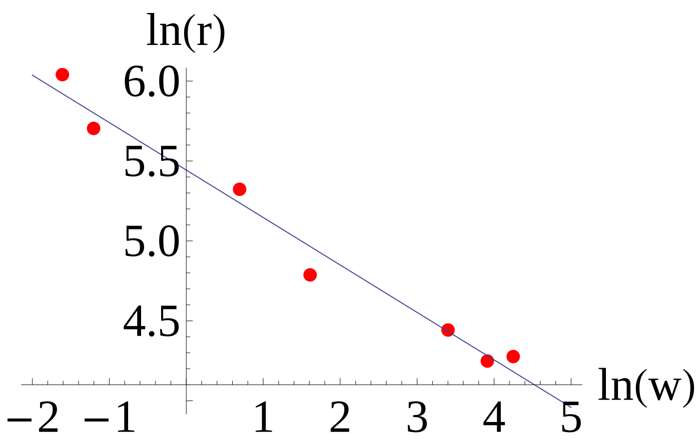

This is a good looking fit. We can now use the following equation to answer questions about this relationship:

$$\ln(r) = -0.296876 \ln(w) + 5.4433$$

In fact, let's take the time to fold this back up to a power form, since that was originally what we wanted. The first step is to write 5.4433 as a logarithm

$$5.4433 = \ln(a)$$
$$e^{5.4433} = e^{\ln(a)}$$
$$231.204 = a$$

We do this because it will allow us to collect the logarithms. Can you identify the logarithm rule we used at each step?

$$\ln(r) = -0.296876 \ln(w) + 5.4433$$
$$\ln(r) = -0.296876 \ln(w) + \ln(231.204)$$
$$\ln(r) = \ln\left(w^{-0.296876}\right) + \ln(231.204)$$
$$\ln(r) = \ln\left(231.204\, w^{-0.296876}\right)$$
$$\exp(\ln(r)) = \exp\left(\ln\left(231.204\, w^{-0.296876}\right)\right)$$
$$r = 231.204\, w^{-0.296876}$$

If you check the answer in the text for Problem 55 in Section 2.2, you will see that they got $r = 231.204\, w^{-0.297}$, which is what we have arrived at. And we can understand each step in the process of how we got it, which is important.

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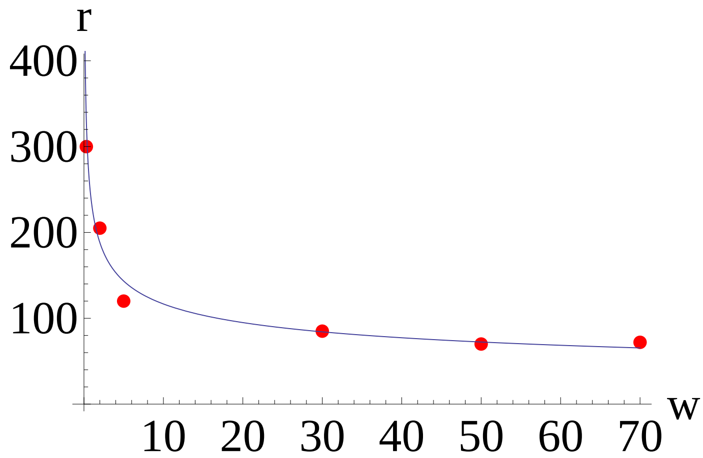

If we wanted to know the pulse for a 450 kg horse, we could use our approximation to get it. Our model predicts a pulse of 38 beats per minute.

$r = 231.204\,(450)^{-0.296876} = 37.67$