1101 Calculus I Chapter 1 Review

This is a set of notes identifying the main concepts of Chapter 1. You should understand and be able to work with all these concepts in a variety of situations; **this is not meant to be comprehensive!** You should understand these concepts, and be able to apply them to examples and use them to solve problems, like the ones we worked in class and on assignments, and like the concept checks and true or false problems from the review.

I will not be providing similar notes for review for later sections. It is more beneficial to you if you create these notes yourself as we go along, adding detail (examples, worked problems, graphs, etc) where you feel appropriate.

Section 1.1 Four Ways to Represent a Function

What is a function? A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

The **range** B is the set of all possible values of $f(x)$, when x varies over the entire **domain** A.

Machine View:

$$\underset{\text{(from domain)}}{\underset{\text{input}}{x}} \longrightarrow \boxed{\quad f \quad} \longrightarrow \underset{\text{(in range)}}{\underset{\text{output}}{f(x)}}$$

Representations of a function (the four ways)

1. verbally (words)
2. numerically (table)
3. visually (graph)
4. algebraically (formula)

Vertical Line Test A graph represents a function if every vertical line you can draw intersects the graph only once (this ensures we have exactly one element $f(x)$ for each x).

Symmetry

Even functions satisfy $f(-x) = f(x)$. Geometrically this means the function is symmetric about the y-axis.

Odd functions satisfy $f(-x) = -f(x)$. Geometrically this means the function is symmetric if we rotate 180 degrees about the origin.

NOTE: A function can be either even, or odd, or neither!

Increasing and Decreasing Functions

Increasing A function is increasing on an interval I if $F(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

Decreasing A function is decreasing on an interval I if $F(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

Section 1.2 Mathematical Models

Polynomial Functions A function is called a polynomial if it is of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the a's are constants called the coefficients of the polynomial. The domain of the polynomial is the reals, and the degree of the polynomial is n.

A polynomial of degree one is a linear function: $P(x) = mx + b$.

A polynomial of degree 2 is called quadratic: $P(x) = ax^2 + bx + c$.

A polynomial of degree 3 is a cubic function: $P(x) = ax^3 + bx^2 + cx + d$.

Power Functions A power function is of the form: $f(x) = x^a$. And we have several cases:

$a = n, n$ **is a positive integer** This reduces in essence to the polynomials.

$a = 1/n, n$ **is a positive integer** This is a root function.

$$f(x) = x^{1/n} =^n \sqrt{x}.$$

For $n = 1$ it is the square root.

$a = -1$ This is the reciprocal function. It is a hyperbola with the coordinate axes as its asymptotes.

$$yx = 1.$$

Rational Functions are just the ratio of two polynomials.

$$f(x) = \frac{P(x)}{Q(x)}$$

and has the domain all values x such that $Q(x) \neq 0$.

Algebraic Functions are functions that are constructed using algebraic operations on polynomials. These functions can have extremely bizarre looking graphs!

Trigonometric Functions Know the graphs! $\sin x, \cos x, \tan x$ (2π radians $=$ 360 degrees)

Exponential Functions Exponential functions are of the form $f(x) = a^x$ where a is called the base and is a positive constant. Much more later.

Logarithmic Functions The log functions are the inverse of the exponential functions. Much more later.

Transcendental Functions Any function which is not algebraic is transcendental. The trig, exponential, and their inverses are examples of transcendental functions. A transcendental function is usually characterized by an infinite series expansion, and there are a huge number of transcendental functions.

Section 1.3: New Functions From Old Functions

Translations

Vertical and Horizontal Shifts: Suppose $c > 0$, to obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting: Stretching: If $c > 1$ then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction.

Reflection: The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis, because our ordered pair has changed from $(x, f(x))$ to $(x, -f(x))$.

Suppose $c > 1$, to obtain the graph of

$y = cf(x)$ stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$ compress the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$ compress the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$ stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis

Algebraic Combinations of Functions

Let f and g be functions with domains A and B. Then the functions $f + g, f - g, fg, f/g$ are defined as:

$(f + g)(x) = f(x) + g(x)$ domain $A \cap B$

$(f - g)(x) = f(x) - g(x)$ domain $A \cap B$

$(fg)(x) = f(x)g(x)$ domain $A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $= \{x \in A \cap B | g(x) \neq 0\}$

Graphical Addition We can add functions graphically by by simply adding their corresponding y-coordinates.

Composition of functions

Given two functions f and g, the **composite** function $f \circ g$ (called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is important to note that $f \circ g(x) \neq g \circ f(x)$!!

Graphical Composition To find the composition $f \circ g(a)$ from a graph or table, we can use the following method (if a table is given, create graphs of the functions and apply a best fit line).

From the graph read $g(a)$, and then use the value $g(a)$ as the input to determine $f(g(a))$. You can build a table of values this way, and then plot $f(g(x))$.

Example of Multiple Compositions $(f \circ g \circ h)(x)$

$$f(x) = \frac{x}{x+1},\ g(x) = x^{10},\ h(x) = x+3$$

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) \\ &= f(g(x+3)) \\ &= f((x+3)^{10}) \\ &= \frac{(x+3)^{10}}{(x+3)^{10}+1}\end{aligned}$$

Example of Decomposing using composition Given $L(x) = \cos^2(x+9)$, determine functions f, g, h so you can write $L(x)$ as a composition $(f \circ g \circ h)(x)$ (that is, $L(x) = (f \circ g \circ h)(x)$).

SOL:

Look at how you compute $L(x)$, and build the functions from that:

Add 9: $h(x) = x + 9$

Take cosine: $g(x) = \cos x$

Square: $f(x) = x^2$

Section 1.5: Exponential Functions

Exponential functions are of the form $f(x) = a^x$ where a is a positive constant.

Increasing, Decreasing or Constant The exponential function $f(x) = a^x$ will either be

Increasing, if $1 < a < \infty$,

Decreasing, if $0 < a < 1$,

Constant, if $a = 1$.

For the increasing exponential, the function will increase more rapidly (as $x \to \infty$) the larger a gets.

All exponential functions pass through the point $(0, 1)$ (unless they are shifted, of course!). If $a \neq 1$ the domain is all reals and the range is $(0, \infty)$.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

$$\begin{aligned}a^{x+y} &= a^x a^y \\ a^{x-y} &= a^x a^{-y} \\ (a^x)^y &= a^{xy} \\ (ab)^x &= a^x b^x\end{aligned}$$

Applications of Exponential Functions: Population Growth Suppose that we know a population doubles every hour. Say the population is $p(t)$, and t is the time in hours. If our initial population is p_0, then we have:

$$\begin{aligned}
p(0) &= p_0 \\
p(1) &= 2p_0 \\
p(2) &= 2p(1) = 2^2 p_0 \\
p(3) &= 2p(2) = 2^3 p_0 \\
\\
p(t) &= 2^t p_0
\end{aligned}$$

And we see how the exponential function enters the field of population growth.

Q: What if the population tripled every hour, how would that change our final answer? A: $p(t) = 3^t p_0$.

Applications of Exponential Functions: Half-Life The half-life is defined for radioactive compounds to be the time it takes for the amount to decay into half that amount.

The half-life for strontium-90 (^{90}Sr) is 25 years. If we have 24mg of ^{90}Sr, find an expression for amount left after t years.

$$\begin{aligned}
m(0) &= 24 \\
m(25) &= \frac{1}{2}(24) \\
m(50) &= \frac{1}{2}m(25) = \frac{1}{2^2}m(0) \\
m(75) &= \frac{1}{2}m(50) = \frac{1}{2^3}m(0) \\
\\
m(t) &= \frac{1}{2^{t/25}}(24) = 24 \cdot 2^{-t/25}
\end{aligned}$$

There was some talk about the number e More on that later with Logarithms and natural Logarithms.

Section 1.6: Inverse Function and Logarithms

Not all functions have inverses. Recall that a function is defined as a rule f that assigns to each element x in a set A **exactly** one element, called $f(x)$ in a set B.

This requirement that a function have exactly one element imposes a condition on whether or not a function f has an inverse function.

Definition A function f is called a **one-to-one** function if it never takes on the same value twice, that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

You can check whether or not a function is one-to-one from the graph by using the horizontal line test.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by:

$$f^{-1}(y) = x \longleftrightarrow f(x) = y$$

for any y in B.

The letter x is usually reserved for the independent variable, so we can rewrite our definition as:

$$f^{-1}(x) = y \longleftrightarrow f(y) = x$$

when we are talking about the inverse function.

Cancellation Equations

$$f^{-1}(f(x)) = x \text{ for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \text{ for every } x \text{ in } B$$

How to find the inverse function of a one-to-one function

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$

Inverse from a Graph Since $f(a) = b$ iff $f^{-1}(b) = a$, the point (a, b) is on the graph of f iff the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from reflecting about the line $y = x$:

Technique The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. Therefore, it has an inverse, f^{-1} which is called the **logarithmic function with base** a and is denoted $\log_a$. Using our definition of inverse functions, we have

$$\log_a(x) = y \longleftrightarrow a^y = x.$$

So, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x:

$$\log_{10} 0.0001 = -4 \text{ since } 10^{-4} = 0.0001$$

The cancellation equations can be written for the logarithmic and exponential functions as:

$$\log_a(a^x) = x \text{ for every } x \in (-\infty, \infty)$$
$$a^{\log_a(x)} = x \text{ for every } x \in (0, \infty)$$

Laws of Logarithms If x and y are positive numbers, then

$$\begin{aligned}\log_a(xy) &= \log_a x + \log_a y \\ \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \\ \log_a(x^r) &= r\log_a x \text{ where r is any real number}\end{aligned}$$

Natural Logarithms

The **natural logarithm** has as base the number e. It is usually written

$$\log_e x = \ln x.$$

The defining properties of the natural logarithm are:

$$\ln x = y \longleftrightarrow e^y = x$$

Cancellation equations:

$$\begin{aligned}\ln(e^x) &= x \qquad x \in (-\infty, \infty) \\ e^{\ln x} &= x \qquad x > 0\end{aligned}$$

If we set $x = 1$, we see that $\ln e = 1$.

Theorem For any positive number $a (a \neq 1)$, we have

$$\log_a x = \frac{\ln x}{\ln a}.$$

Understand the proof!

Example: ^{90}Sr decay We looked at the example of radioactive decay, and found how the mass of Strontium 90 decayed as a function of time t (years). Now we can find the inverse function and interpret what it means for this case.

The equation we determined before was:

$$m = f(t) = 24 \cdot 2^{-t/25}$$

Take the natural logarithm of both sides:

$$\begin{aligned}\ln m &= \ln(24 \cdot 2^{-t/25}) \\ \ln m &= \ln 24 + \ln 2^{-t/25} \\ \ln m &= \ln 24 - \frac{t}{25}\ln 2 \\ \frac{t}{25}\ln 2 &= \ln 24 - \ln m = \ln\left(\frac{24}{m}\right) \\ t &= \frac{25}{\ln 2}\ln\left(\frac{24}{m}\right)\end{aligned}$$

So the inverse function is

$$t = f^{-1}(m) = \frac{25}{\ln 2}\ln\left(\frac{24}{m}\right)$$

which gives the time it takes for the mass to decay to m milligrams.