Calculus I Handout: Curves and Surfaces in $\mathbb{R}^3$

Up until now, everything we have worked with has been in two dimensions. But we can extend the concepts of calculus to three dimensions (and beyond!). Why would we want to do this? Well, we can think of functions that depend on more than one variable. For example, the cost of producing a pencil may depend on the price of the wood chips used in the pencil, the cost of labour, the cost of running a factory, and many other things. There may be no relationship between these things, and so eliminating some of the variables by finding relationships between them could be difficult (if not impossible) to do. Sometimes, a relationship between multiple variables can be determined–for example, Ohm's Law in electric circuits is $V = IR$, where voltage V is determined in terms of the product of the the current I and resistance R in the circuit. We express this dependence mathematically by writing $V(I, R)$.

The study of calculus that involves functions of more than one variable is called *multivariable calculus*, and it is studied in depth in Calculus III. However, we want to introduce some of the basic concepts of multivariable calculus to you now. We will focus on three dimensions, $\mathbb{R}^3$. These concepts will include

- three dimensional coordinate systems,
- surfaces in $\mathbb{R}^3$ (explicit representation),
- traces, contour plots,
- space curves in $\mathbb{R}^3$ (parametric),
- partial derivatives,
- introduction to extrema in $\mathbb{R}^3$.

We will begin by reviewing some of the concepts we have learned in $\mathbb{R}^2$, and then we shall extend these concepts to $\mathbb{R}^3$.

1 Curves in $\mathbb{R}^2$

An ordered pair (x, y) is needed to locate a point in the plane. The set of all ordered pairs of real numbers is the Cartesian product $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$.

In Fig. 1 the two dimensional xy-coordinate system is shown. This is an orthogonal coordinate system, since the angle between the x-axis and y-axis is $\pi/2$ radians (or 90 degrees). A point is designated by the ordered pair (x, y). The coordinate lines $x = 0$ (y-axis) and $y = 0$ (x-axis) split $\mathbb{R}^2$ into four quadrants.

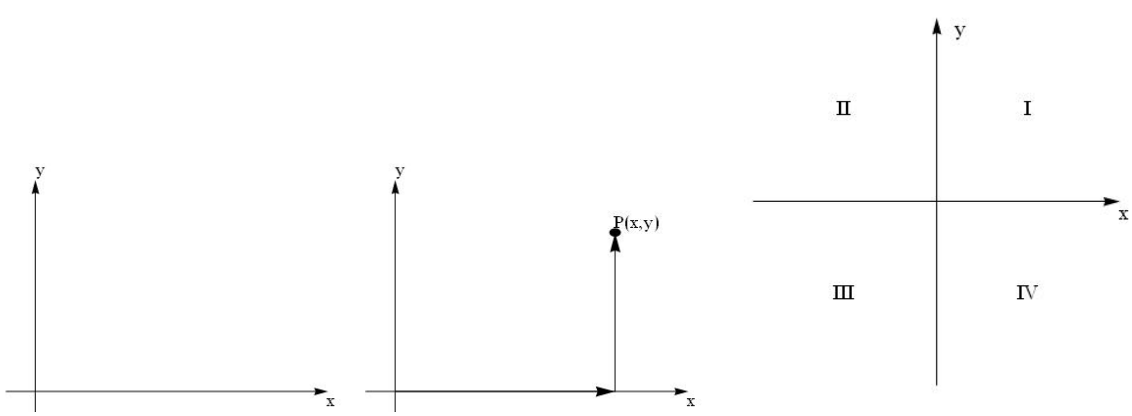
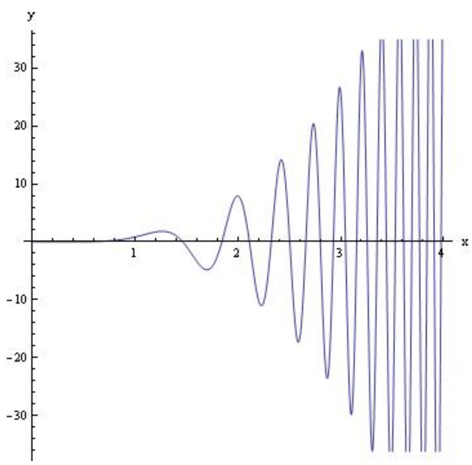


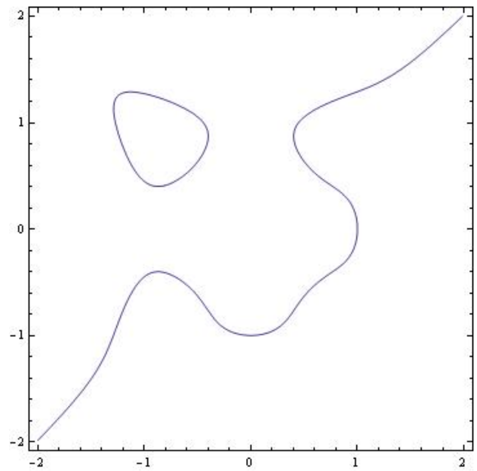
Figure 1: An orthogonal two dimensional coordinate system, locating a point in the system, and the four quadrants.

We have seen three different ways to describe a function in $\mathbb{R}^2$. They are:
Explicit function: $y = f(x)$
Implicit function: $F(x, y) = 0$
Parametric function: $x = f(t), y = g(t), \quad \alpha \leq t \leq \beta$

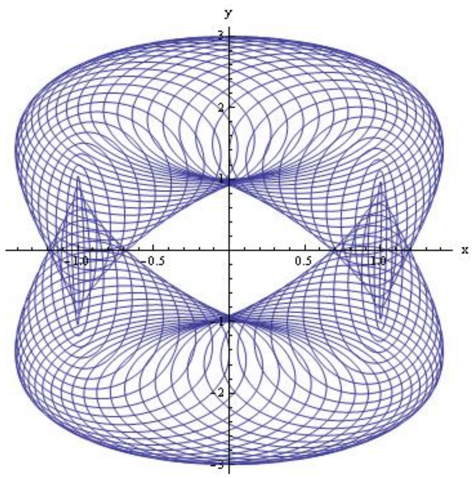
In Fig. 2 there are plots of the three types of functions, and the *Mathematica* commands which generated the plots. As we have noted before, implicit functions and parametric functions can produce wonderfully beautiful curves.




```
Plot[x^3 Sin[x^3], {x, 0, 4},
     AxesLabel -> {"x", "y"},
     AspectRatio -> 1]
```




```
ContourPlot[x^11 - y^11 + 10 x^2 y^2 == 1,
     {x, -2, 2}, {y, -2, 2},
     AxesLabel -> {"x", "y"},
     AspectRatio -> 1,
     PlotPoints -> 100]
```




```
ParametricPlot[{Cos[t] - Cos[80t] Sin[t], 2 Sin[t] - Sin[80t]},
     {t, -Pi, Pi}, PlotPoints -> 400,  AxesLabel -> {"x", "y"},
     AspectRatio -> 1]
```

Figure 2: An example of plotting explicit function: $y = x^3 \sin x^3$, implicit function: $x^{11} - y^{11} + 10x^2y^2 = 1$, and a parametric function: $x = \cos t - \cos 80t \sin t, y = 2 \sin t - \sin 80t, \ -\pi \leq t \leq \pi$ using *Mathematica.*

What we have in all cases is that the graph of an equation (explicit, implicit or parametric) involving x and y is a curve in $\mathbb{R}^2$. We now turn our attention to extending these ideas to coordinate systems of higher dimension. We shall see that instead of only having curves, we will get curves and *surfaces* in $\mathbb{R}^3$.

2 Coordinates, Curves and Surfaces in $\mathbb{R}^3$

An ordered triple (x, y, z) is needed to locate a point in three dimensional space. The set of all ordered triples of real numbers is the Cartesian product $\mathbb{R}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\}$.

In Fig. 3 the three dimensional xyz-coordinate system is shown. This is an orthogonal coordinate system, since the angle between all the coordinate axes is $\pi/2$ radians. It is also known as a *right hand coordinate system.* since it satisfies the *right hand rule.*

The right hand rule is as follows: With your right hand, point your fingers along the positive x-axis. Now, curl them so that the tips of your fingers point in the direction of the positive y-axis. Extend your thumb, and it points in the direction of the positive z axis. If your thumb points in the direction of the negative z-axis, you have a *left hand coordinate system.*

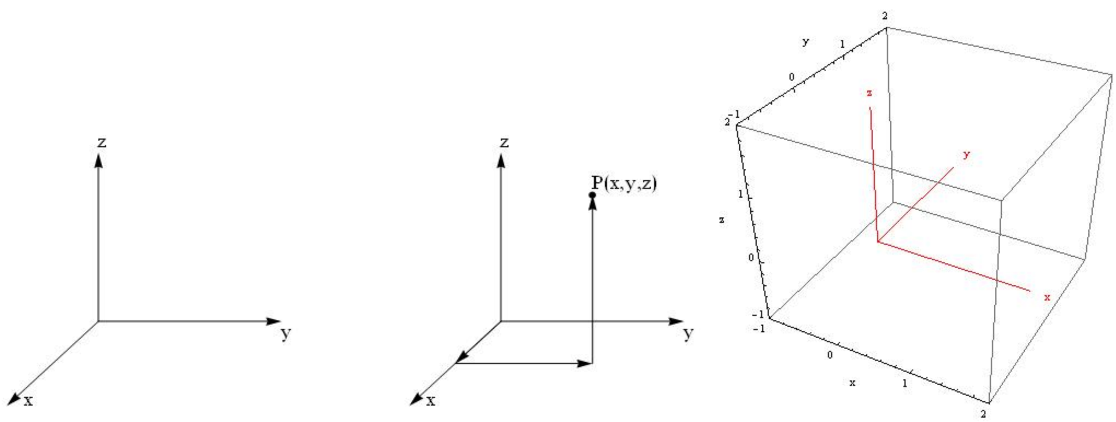


FIGURE 3: THE ORTHOGONAL RIGHT HANDED THREE DIMENSIONAL xyz-COORDINATE SYSTEM. THINK OF THE x-AXIS AS COMING OUT OF THE PAGE TOWARDS YOU. ON THE LEFT IS HOW *Mathematica* DRAWS ITS AXES BY DEFAULT. NOTICE THAT y IS GOING INTO THE PAGE AND x IS COMING OUT OF THE PAGE.

The three coordinate axes determine the three *coordinate planes* (Fig. 4). The coordinate planes are determined by setting one of the coordinates to zero. Hence, the xy-plane is given by $z = 0$; the zy-plane is given by $x = 0$; and the zx-plane is given by $y = 0$.

Notice in Fig. 5 that the coordinate planes divide $\mathbb{R}^3$ into eight octants. In two dimensions, the coordinate lines divide $\mathbb{R}^2$ into four quadrants.

Problem 1. Sketch the points $(3, 4, 2)$, $(-4, 4, 4)$, $(0, -3, 4)$ on coordinate axes.

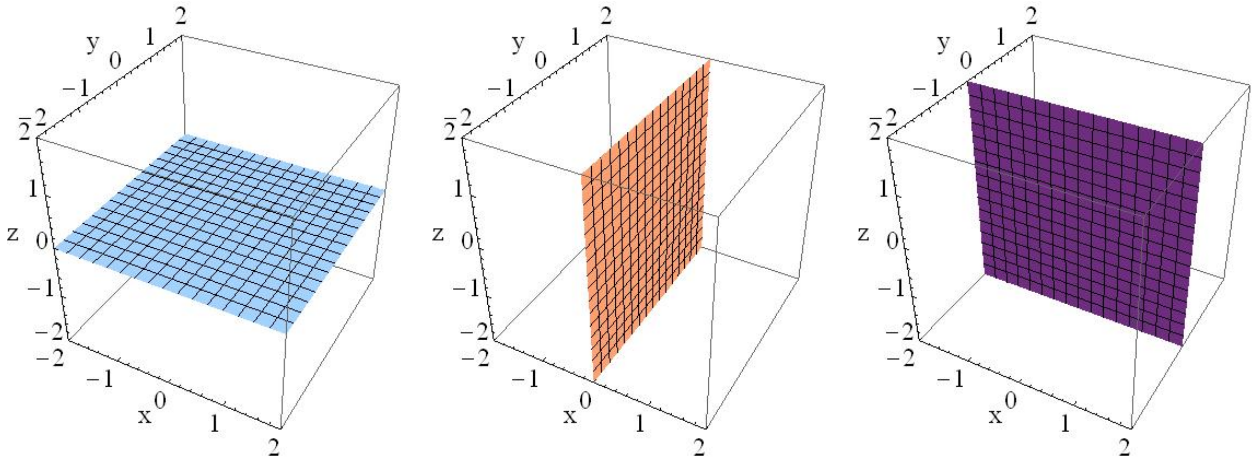
Problem 2. Sketch the planes $x = -3$ and $y = 4$ on a single set of coordinate axes.

2.1 Surfaces in $\mathbb{R}^3$

A *surface* in $\mathbb{R}^3$ can be generated in different ways. You could imagine that we will be able to generate explicit representations of surfaces, implicit representations of surfaces, and parametric representations of surfaces.

Explicit functions are easily constructed in $\mathbb{R}^3$, and can easily be visualized by using the *Mathematica* command `Plot3D`.

Parametric functions are very interesting and important in multivariable calculus. There is also a *Mathematica* command



xy-plane ($z = 0$) zy-plane ($x = 0$) zx-plane ($y = 0$)

Figure 4: The coordinate planes.

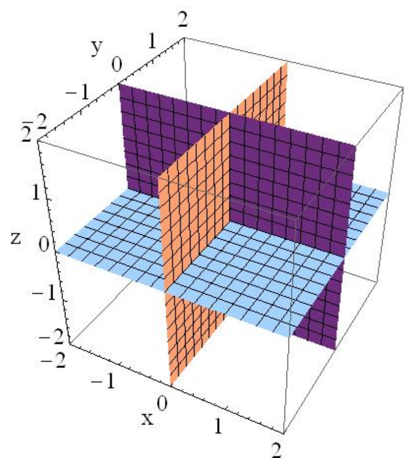


Figure 5: The coordinate planes divide $\mathbb{R}^3$ into eight octants.

(`ParametricPlot3D`) which allows one to plot a parametric function in $\mathbb{R}^3$.

Although you might guess that one should exist, there is no command `ImplicitPlot3D` in *Mathematica*! Typically, `Plot3D` and `ParametricPlot3D` will allow you to plot any function you desire.

Although parametric functions are useful and interesting, we will only be considering the explicit representation of a surface in $\mathbb{R}^3$. Parametric functions in $\mathbb{R}^3$ are studied in depth in Calculus III.

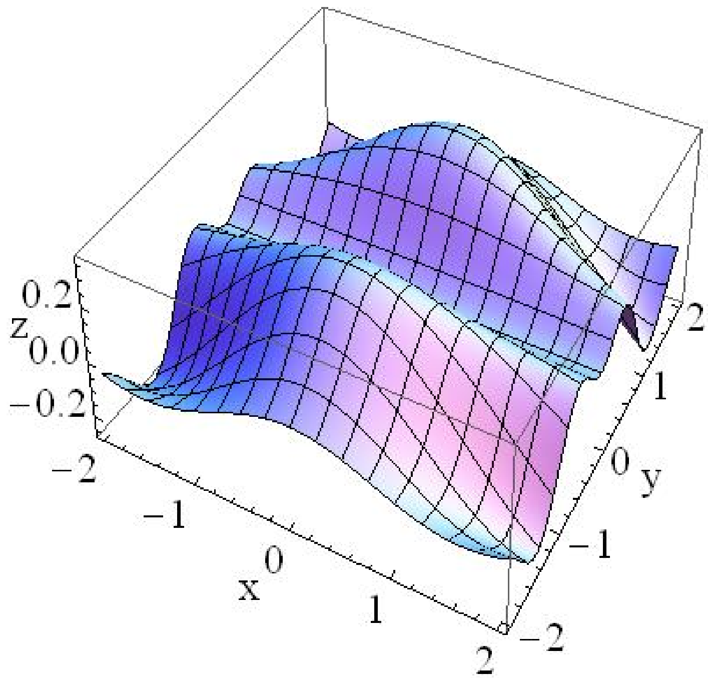
2.1.1 Explicit Representation of Surfaces in $\mathbb{R}^3$

In $\mathbb{R}^2$, a function could be explicitly represented as $y = f(x)$. Given x, we can use f to determine y in the ordered pair (x, y).

Extending this idea to $\mathbb{R}^3$, we can write $z = f(x, y)$, and if we are given any ordered pair (x, y), we can use f to determine

z in the ordered triple (x, y, z).

For example, consider the function $z = y^2 \cos(xy)e^{-y^2}$. Given (x, y), we can determine the value of z which goes with that ordered pair. We could imagine constructing a table of values, and graphing many dots over the xy-plane by hand. Thankfully, there is an easier method. We can plot this function using the *Mathematica* command `Plot3D`. Figure 6 contains a plot of this function.




```
Plot3D[y^2 Cos[x*y]Exp[-y^2],
  {x, -2, 2}, {y, -2, 2}, AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z  "}, PlotPoints -> 40]
```

Figure 6: The surface given by the explicit function $z = y^2 \cos(xy)e^{-y^2}$.

An important aspect of working in $\mathbb{R}^3$ is being able to sketch and describe surfaces. Does the surface look like a sphere? A cone (straight sides)? A paraboloid (curved sides)? Does the surface open up, down, or does it open along an axis other than the z-axis? What do I mean by all these questions?

Notice that the surface in Fig 6 is drawn with grid lines. One set of grid lines have a fixed y value and are contained in a plane parallel to the xz-plane (cf. Fig. 4). The other grid lines have a fixed x value, and are contained in a plane parallel to the yz-plane.

By examining these grid lines in more detail, we can answer the questions posed above.

2.2 Traces of Surfaces

A trace of a surface is just a fancy way of saying a *cross section*. If we think of a cross section as a slice through something–an idea all the biologists will be familiar with–then we already understand what the concept of trace means mathematically, for these are essentially the same concept.

Consider our previous example, $z = y^2 \cos(xy)e^{-y^2}$. First, let's take cross sections that have a fixed value of x. We show this graphically in Fig. 7.

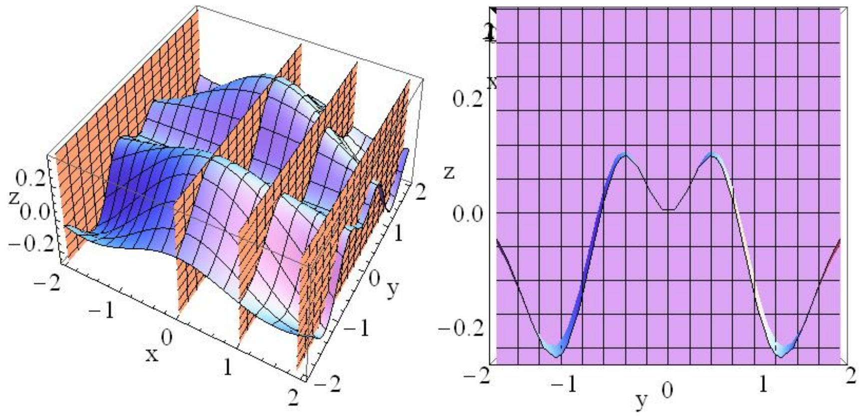


FIGURE 7: THE SURFACE GIVEN BY THE EXPLICIT FUNCTION $z = y^2 \cos(xy)e^{-y^2}$, AND HOW THE TRACE IN THE yz-PLANE IS CONSTRUCTED VIA CROSS SECTIONS. ON THE RIGHT, THE VIEWPOINT HAS BEEN CHANGED SO WE ARE LOOKING AT THE CURVE FROM THE END ON, SO WE SEE ESSENTIALLY THE CROSS SECTION THAT RESULTS WHEN THE PLANE AT $x = 2$ IS USED TO SLICE THE SURFACE.

What we need to be able to do is figure out analytically what the traces are. If $x = k$, where k is a constant, we get the function in $\mathbb{R}^2$

$$z = y^2 \cos(ky)e^{-y^2},$$

which means we have a function in the zy-plane governed by a parabola, cosine and decaying exponential. By varying the constant k we will get different curves–so the trace is really a *family of curves*.

Second, let's take cross sections that have a fixed value of y. We show this graphically in Fig. 8.

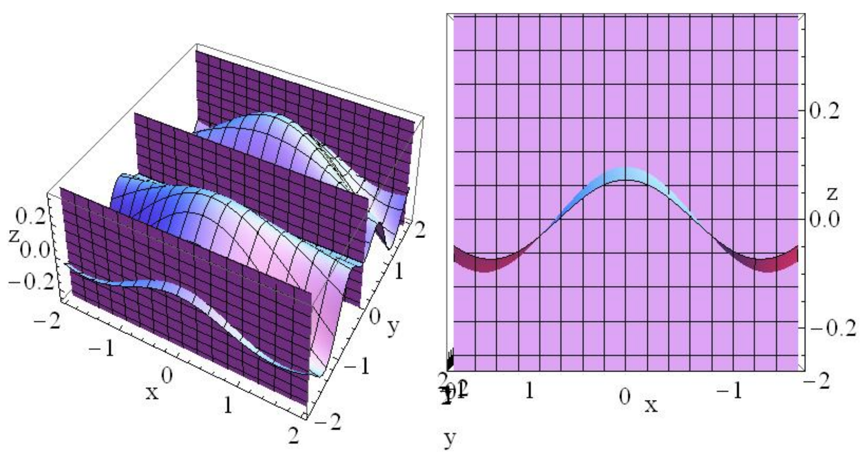


FIGURE 8: THE SURFACE GIVEN BY THE EXPLICIT FUNCTION $z = y^2 \cos(xy)e^{-y^2}$, AND HOW THE TRACE IN THE xz-PLANE IS CONSTRUCTED VIA CROSS SECTIONS. ON THE RIGHT, THE VIEWPOINT HAS BEEN CHANGED SO WE ARE LOOKING AT THE CURVE FROM THE END ON, SO WE SEE ESSENTIALLY THE CROSS SECTION THAT RESULTS WHEN THE PLANE AT $y = -2$ IS USED TO SLICE THE SURFACE.

If $y = k$, where k is a constant, we get the function in $\mathbb{R}^2$

$$z = k^2 \cos(xk)e^{-k^2},$$

which means we have a cosine function in the zx-plane. By varying the constant k we will get a family of curves.

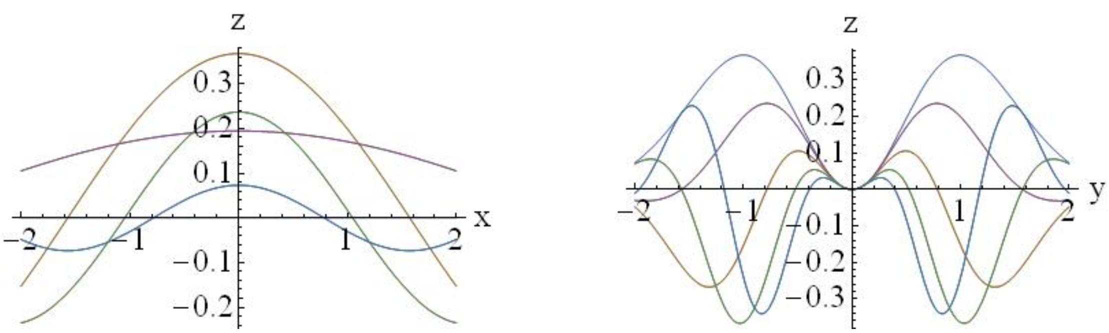
There is a third trace, one in the xy-plane, and it is usually called a *contour plot.* We will learn about contour plots shortly. Another way of thinking of the trace is as a *projection* of the surface onto the coordinate planes.

How to calculate the trace of $z = f(x, y)$

If you want the trace in the zx-plane, you will set $y = k$ and the trace is given by the family of curves $z = f(x, k)$.

If you want the trace in the zy-plane, you will set $x = k$ and the trace is given by the family of curves $z = f(k, y)$.

For the previous example, we can more easily plot the traces by simply plotting the family of curves. This is done in Fig. 9.




```
list = Table[k^2 Cos[x*k]Exp[-k^2],
  {k, -2, 2, 1/2}]
Plot[Evaluate[list], {x, 0, 5 Pi}]
```

```
list = Table[y^2 Cos[k*y]Exp[-y^2],
  {k, -4, 4}]
Plot[Evaluate[list], {y, -2, 2}]
```

FIGURE 9: THE TRACES IN THE zx-PLANE (LEFT) AND THE zy-PLANE (RIGHT) FOR THE SURFACE $z = y^2 \cos(xy)e^{-y^2}$.

Example 1. Sketch and describe the traces in the zx-plane and the zy-plane for the surface $z = x^2 - y^2$.

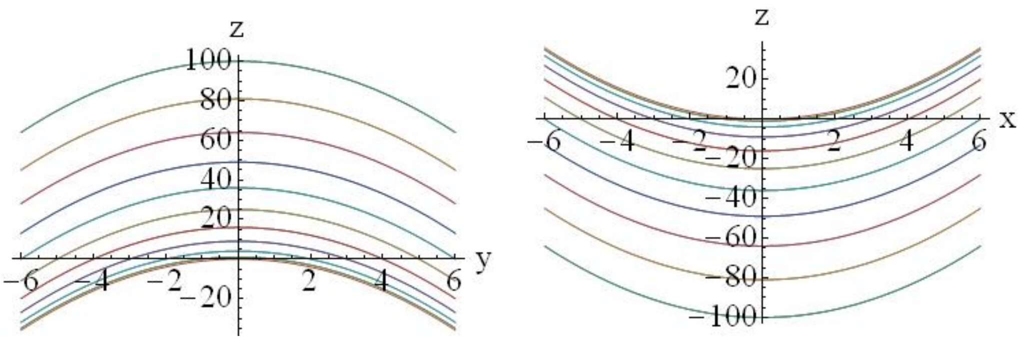
Solution The trace in the zx-plane is given by the family of curves $z = x^2 - k^2$.

The trace in the zy-plane is given by the family of curves $z = k^2 - y^2$.

Sketches of these traces are given in Fig. 10. In the zx-plane we have a family of curves which are parabolas that open up, and in the zy-plane we have a family of curves which are parabolas which open down.

2.3 Contour Plots of Surfaces

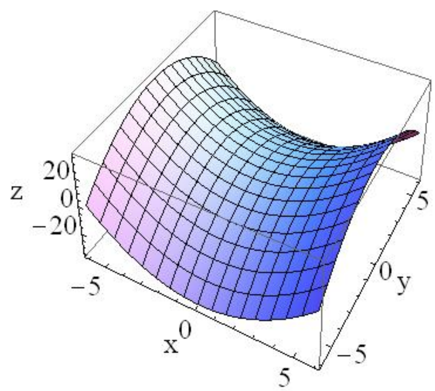
A contour plot of a surface is similar to a trace, in that it represents intersections of the surface with planes. For a contour plot, the planes of intersection are parallel to the xy-coordinate plane.




```
list = Table[k^2 - y^2, {k, -10, 10}]
Plot[Evaluate[list], {y, -6, 6}]
```

```
list = Table[x^2 - k^2, {k, -10, 10}]
Plot[Evaluate[list], {x, -6, 6}]
```

FIGURE 10: THE TRACES IN THE zx-PLANE (RIGHT) AND THE zy-PLANE (LEFT) FOR THE SURFACE $z = x^2 - y^2$.




```
Plot3D[x^2 - y^2, {x, -6, 6}, {y, -6, 6}]
```

FIGURE 11: THE SURFACE $z = x^2 - y^2$.

The traces help us sketch the function, but the contour plots help us determine extrema for the function, and also give us another way of representing the function without resorting to a three dimensional sketch.

Figure 14 shows how planes intersecting the surface create lines with constant z values.

Contour plots are important since the lines in a contour plot join points on the surface with the same height. When you move to a different line, you move to either a larger or smaller value of the function $z = f(x, y)$.

Since contour plots are so important, *Mathematica* has a special command to generate them for a function. Figure 13 shows the contour plot for the function $z = y^2 \cos(xy)e^{-y^2}$. The regions with higher z values are lighter.

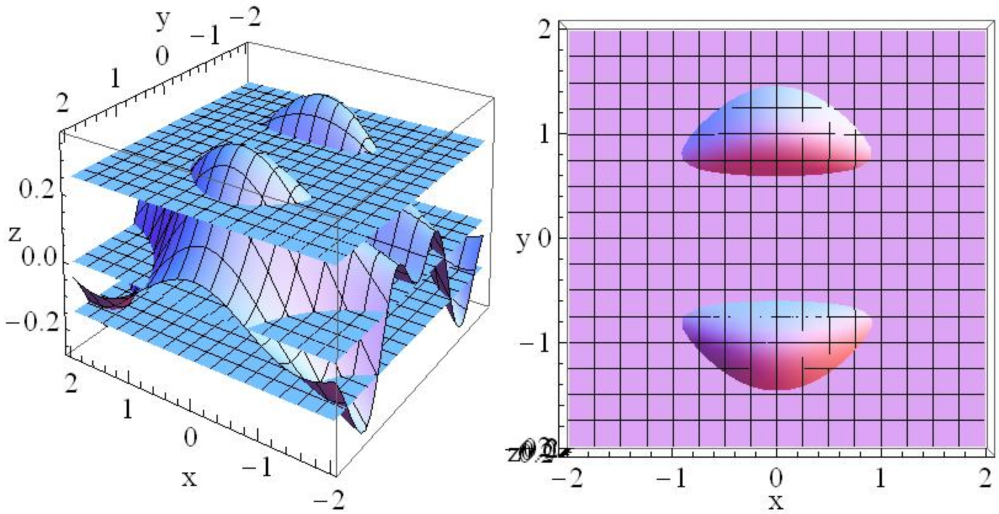
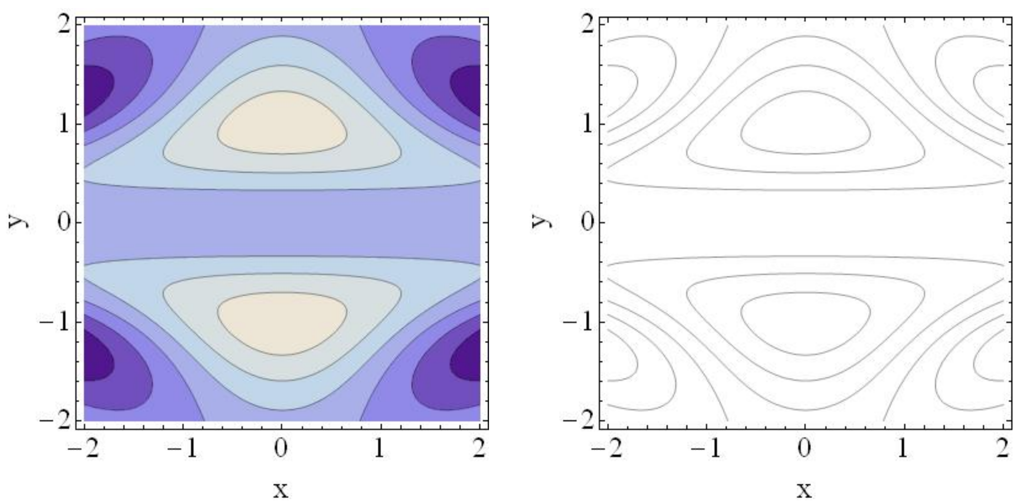


FIGURE 12: THE SURFACE GIVEN BY THE EXPLICIT FUNCTION $z = y^2 \cos(xy)e^{-y^2}$, AND HOW THE TRACE IN THE xy-PLANE (OR CONTOUR PLOT) IS CONSTRUCTED VIA CROSS SECTIONS. ON THE RIGHT, THE VIEWPOINT HAS BEEN CHANGED SO WE ARE LOOKING AT THE CURVE FROM THE TOP ON, SO WE SEE ESSENTIALLY THE CROSS SECTION THAT RESULTS WHEN THE PLANE AT $z = 0.25$ IS USED TO SLICE THE SURFACE.




```
ContourPlot[y^2 Cos[x*y]Exp[-y^2],
  {x, -2, 2}, {y, -2, 2},
  PlotPoints -> 40,
  FrameLabel -> {"x", "y"}]
```

```
ContourPlot[y^2 Cos[x*y]Exp[-y^2],
  {x, -2, 2}, {y, -2, 2},
  PlotPoints -> 40,
  FrameLabel -> {"x", "y"},
  ContourShading -> False]
```

FIGURE 13: THE CONTOUR PLOT FOR $z = y^2 \cos(xy)e^{-y^2}$, WITH AND WITHOUT SHADING.

Problem 3. Sketch and describe the traces in the zx-plane and the zy-plane for the surface $z = 1 - x^2 - y^2$. Also create a contour plot of the function.

Problem 4. Sketch and describe the traces in the zx-plane and the zy-plane for the surface $x^2 + 4y^2 + z^2 = 1$. (For these traces you will want to use `ImplicitPlot`).

Problem 5. Sketch and describe the traces in the zx-plane and the zy-plane for the surface $z = \cos(xy)e^{(-x^2-y^2)/10}$. Also generate a contour plot.

2.4 Space Curves in $\mathbb{R}^3$

The way to generate a curve in $\mathbb{R}^3$ (typically called a *space curve*) is to use a parametric representation of the curve. A space curve in $\mathbb{R}^3$ can be represented by

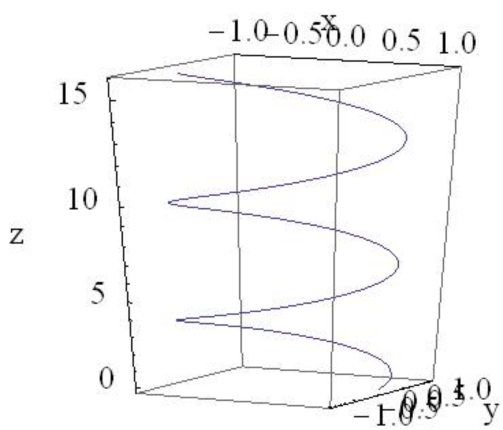
$$x = f(t), y = g(t), z = h(t), \quad \alpha \le t \le \beta.$$

As we vary t, we will sweep out the curve that passes through the points $(f(t), g(t), h(t))$. Notice how this is a simple extension of the concept of a parametric function in $\mathbb{R}^2$.

For example, consider the curve

$$x = \cos t, y = \sin t, z = t, \quad 0 \le t \le 5\pi.$$

This space curve is a helix, and it is plotted in Fig. 14.

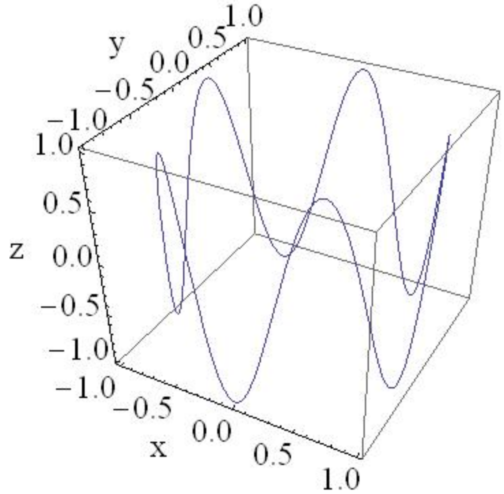



```
ParametricPlot3D[{Cos[t], Sin[t], t}, {t, 0, 5Pi},
  AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z"}]
```

FIGURE 14: THE SPACE CURVE $x = \cos t, y = \sin t, z = t, \quad 0 \le t \le 5\pi$.

Another example of a space curve is

$$x = \cos t, y = \sin t, z = \sin 5t, \quad 0 \le t \le 2\pi.$$




```
ParametricPlot3D[{Cos[t], Sin[t], Sin[5t]}, {t, 0, 2Pi},
  AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z"}]
```

FIGURE 15: THE SPACE CURVE $x = \cos t, y = \sin t, z = \sin 5t, \quad 0 \le t \le 2\pi$.

Problem 6. This problem is a *Mathematica* investigation problem.

Investigate the space curves

$$x = \cos t, y = \sin t, z = \sin at, \quad 0 \le t \le 2\pi,$$

for $a = 0, 1, 2, 3, ..., 20$. What do you notice about the curve as a changes? What do you notice about how *Mathematica* draws the curve as a gets large? Can you think of a reason why the things you see happen? Can you think of a way to improve the sketch for large a?

Space curves are very difficult to draw by hand. We almost always have to resort to using a computer. But even then we have to keep our wits about us.

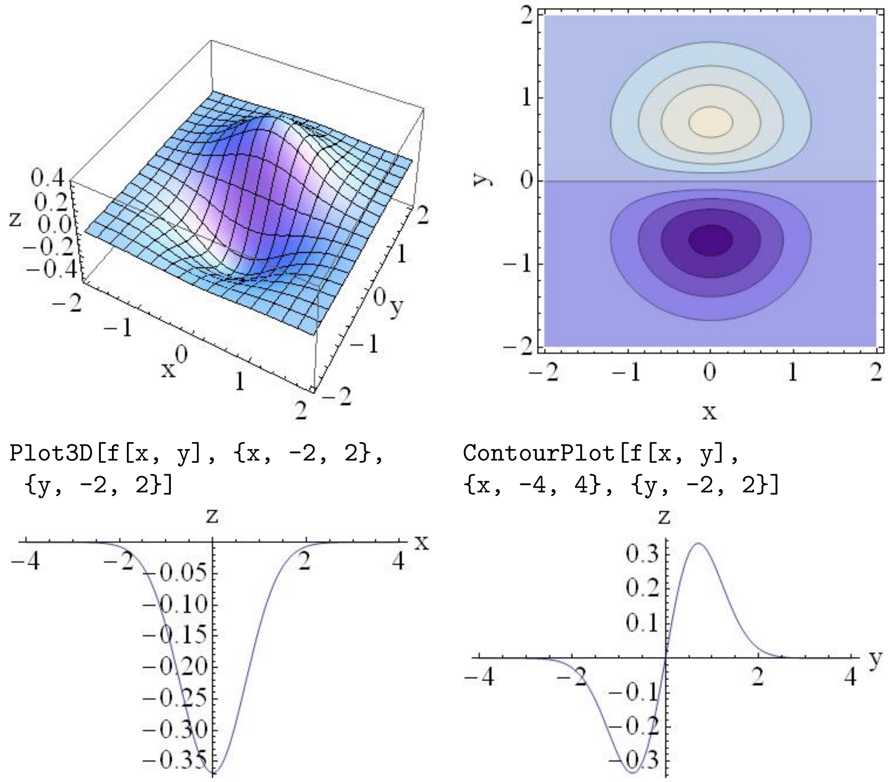
3 Partial Differentiation

Once we have the idea of a surface in $\mathbb{R}^3$, we can think of derivatives of the function. This is where the Leibniz notation of derivative really begins to help us, and the "prime" notation is essentially discarded.

In Fig. 16 we have a surface in $\mathbb{R}^3$ that represents the function $z = f(x, y) = ye^{-x^2-y^2}$.

This graphical representation contains a grid on the surface consisting of paths in the x direction and the y direction. These paths are, respectively, the traces in the zy-plane and the zx-plane.

If we are moving along a path in the x-direction (trace in the zy-plane), then we are holding y fixed at a value that picks out the path we are on. For example, if we look at the path in the x direction that is represented by $f(x, -1)$, we get a path which is shown in Fig. 16.



`Plot3D[f[x, y], {x, -2, 2}, {y, -2, 2}]`

`ContourPlot[f[x, y], {x, -4, 4}, {y, -2, 2}]`

`Plot[f[x, -1], {x, -4, 4}]`

`Plot[f[1/2, y], {x, -4, 4}]`

Figure 16: Top Left: The surface $z = f(x,y) = ye^{-x^2-y^2}$. Top Right: contour plot. Bottom Left: The path $z = f(x,-1) = -e^{-x^2-1}$ in the x direction. Bottom Right: The path $z = f(1/2,y) = ye^{-y^2-1/4}$ in the y direction.

We could certainly take the derivative of the function $z = f(x,-1)$ with respect to x using the techniques from single variable calculus. We could do the same thing for a path in the y direction, and then find the derivative with respect to y. This leads us to the concept of partial derivative.

There is a new notation for derivative in multivariable calculus, which represents the fact that one of the variables is held constant during the derivative process. We define the partial derivatives of the function $z = f(x,y)$ to be

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}, \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

Note that since our function f was a function of two variables, we have two first partial derivatives.

To find the partial derivative $\dfrac{\partial f}{\partial x}(x,y)$ we treat y as a constant and differentiate with respect to x, using all our usual derivative rules (product rule, quotient rule, chain rule, etc.). Similarly, we find $\dfrac{\partial f}{\partial y}(x,y)$ by treating x as a constant and differentiating with respect to y.

The new symbol ∂ reminds us that we are holding one of the variables fixed while we take the derivative.

Warning: This is different from implicit differentiation of an implicit function $F(x,y) = 0$ where we would think of y as a function of x. Although $z = f(x,y)$ and $F(x,y) = 0$ may look similar, they represent two completely different things;

$z = f(x, y)$ is a surface in $\mathbb{R}^3$, but $F(x, y) = 0$ is an implicit function in $\mathbb{R}^2$.

We can also take higher order partial derivatives. There will be four possible second order partial derivatives:

$$\frac{\partial^2 f}{\partial x \partial x}(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y), \qquad \frac{\partial^2 f}{\partial y \partial y}(x, y) = \frac{\partial^2 f}{\partial y^2}(x, y), \qquad \frac{\partial^2 f}{\partial x \partial y}(x, y), \qquad \frac{\partial^2 f}{\partial y \partial x}(x, y).$$

We get the higher order derivatives like we did for single variable functions, by first calculating the first order derivatives and then differentiating these new functions.

Example Find the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ given $f(x, y) = x^3 y - y^2 x$

Solution

$$\begin{aligned}
f(x, y) &= x^3 y - y^2 x \\
\frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}[x^3 y - y^2 x] \\
&= 3x^2 y - y^2 \\
\frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}[x^3 y - y^2 x] \\
&= x^3 - 2yx \\
\frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}[3x^2 y - y^2] \\
&= 6xy \\
\frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y}[x^3 - 2yx] \\
&= -2x \\
\frac{\partial f}{\partial y \partial x}(x, y) &= \frac{\partial}{\partial y}[3x^2 y - y^2] \\
&= 3x^2 - 2y \\
\frac{\partial f}{\partial x \partial y}(x, y) &= \frac{\partial}{\partial x}[x^3 - 2yx] \\
&= 3x^2 - 2y
\end{aligned}$$

Notice that we have $\frac{\partial f}{\partial x \partial y}(x, y) = \frac{\partial f}{\partial y \partial x}(x, y)$. This equality of the mixed second partial derivative will be true as long as the functions are all continuous (we haven't talked about continuity for a function in $\mathbb{R}^3$, but basically if the function has no sudden jumps it is continuous).

You can perform partial derivatives in *Mathematica* using the command `D`. This provides you an excellent method of checking your derivatives:

```
f[x_, y_] = x^3 y - y^2 x
D[f[x, y], x]
D[f[x, y], y]
D[f[x, y], x, x]
D[f[x, y], y, y]
```

```
D[f[x, y], x, y]
D[f[x, y], y, x]
```

Problems 7–10 Find the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ given $f(x, y)$. You can check your answer using *Mathematica*.

Problem 7. $f(x, y) = xy$.

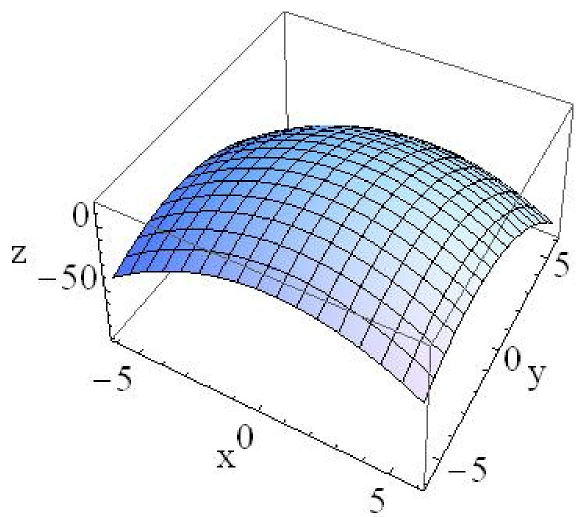
Problem 8. $f(x, y) = \cos(xy)$.

Problem 9. $f(x, y) = \sqrt{x^2 + y^2}$.

Problem 10. $f(x, y) = \tan(y/x)$.

4 Extrema in $\mathbb{R}^3$

Let's look at the function $z = f(x, y) = -x^2 - y^2 + x - 4y + 12$. It is sketched in Fig 17.




```
f[x_, y_] = -x^2 - y^2 + x - 4y + 12
Plot3D[f[x, y], {x, -6, 6}, {y, -6, 6}, AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z     "}, PlotPoints -> 20]
```

Figure 17: The surface $z = f(x, y) = -x^2 - y^2 + x - 4y + 12$.

The surface definitely has a local maximum. A local maximum at (x_0, y_0) means if we move away from the point (x_0, y_0) slightly (in any direction!) the function would be less than the value at (x_0, y_0).

For this reason, finding a maximum in $\mathbb{R}^3$ is often called *hill climbing*. If you start from any point on the surface, hill climbing will lead you to a local maximum (but with no guarantee that you found is a global maximum). Mathematically, you are always moving in the direction where the partial derivatives are increasing most rapdily until you arrive a point where the partial derivatives are both zero.

Therefore, at a local extrema, the first partial derivatives must be zero. This is similar to what we say in $\mathbb{R}^2$: *the tangent line must be horizontal, i.e., the derivative equals zero.*

What we must do is solve the two equations

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0$$

simultaneously for (x, y).

$$\begin{aligned} f(x,y) &= -x^2 - y^2 + x - 4y + 12 \\ \frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}[-x^2 - y^2 + x - 4y + 12] \\ &= -2x + 1 \\ \frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}[-x^2 - y^2 + x - 4y + 12] \\ &= -2y - 4 \end{aligned}$$

$$\begin{aligned} -2x + 1 = 0 \\ -2y - 4 = 0 \end{aligned}$$

The solution was easy in this case, and we find $(x, y) = (-1/2, -2)$ is the point where the maximum occurs, and the maximum is $f(-1/2, -2) = 61/4$.

There are tests equivalent to the second derivative test to verify that what you have found is a maximum or a minimum, however, those tests are a bit too involved for us at this time. We will focus instead on getting a sketch (using either `Plot3D` or `ContourPlot`) to determine if we have a max or min, and then determining the point (x, y) for which the extrema occurs.

Example Find and classify any local extrema for the function $z = f(x, y) = x^4 + y^4 - 4xy - 2y - 4x + 12$.

Solution First, we can plot the function. I am going to use two viewpoints, Although the default viewpoint does indicate that we have a minimum, I feel the second viewpoint shows it better. I've also included a contour plot, which definitely indicates a minimum near the origin. All sketches are in Fig. 18.

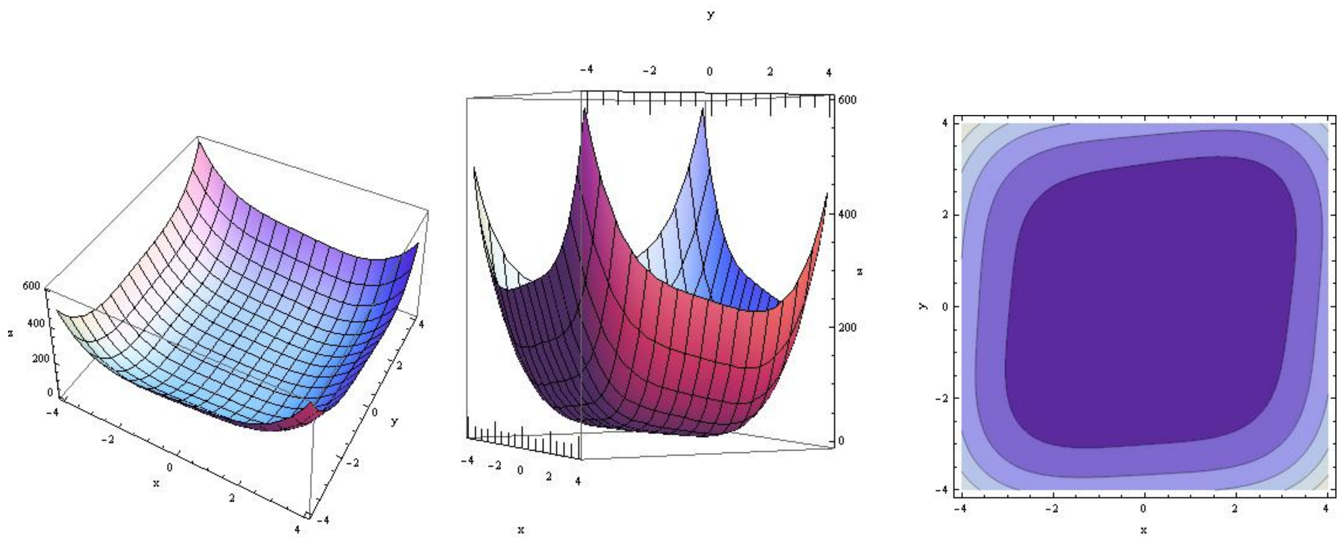
Now we know we are looking for a minimum. We can find the point where the minimum occurs with the help of *Mathematica*:

```
eq1 = D[f[x, y], x] == 0
eq2 = D[f[x, y], y] == 0
NSolve[{eq1, eq2}, {x, y}]
```

The only real valued point here is $(x, y) = (1.304, 1.217)$, and this point seems reasonable based on our sketch. Therefore, the function has a local minimum at $(x, y) = (1.304, 1.217)$ of $f(1.304, 1.217) = 3.087$

Problem 11. Find the value of (x, y) for which the function $z = f(x, y) = e^{4y - y^2 - x^2}$ has a local extrema. Is the extrema a max or a min? Clearly indicate the equations you are solving to find the extrema.

Problem 12. Find the value of (x, y) for which the function $z = -\sqrt{36 - x^2 - y^2 + x + 2y}$ has a local extrema. Is the extrema a max or a min? Clearly indicate the equations you are solving to find the extrema.



```
Plot3D[f[x, y], {x,-4,4},
 {y,-4,4}]
```

```
Plot3D[f[x, y], {x,-4,4},
 {y,-4,4},
 ViewPoint -> {10, -5, 0.1}]
```

```
ContourPlot[f[x, y], {x,-4,4},
 {y,-4,4}]
```

FIGURE 18: THE SURFACE $z = f(x, y) = x^4 + y^4 - 4xy - 2y - 4x + 12$.

5 Final Thoughts

There are a great many extensions of the concepts from single variable calculus to multivariable calculus. If a single integral represents area under a curve, we can imagine that a double integral would represent volume. There are extensions of the Closed Interval Method for finding extrema and The Second Derivative Test for classifying extrema. There are tangent planes rather than tangent lines. And there are other concepts as well, things like vectors, divergence, gradient, surface integrals, line integrals, directional derivatives, and Lagrange multipliers. Multivariable calculus has a direct and significant relationship with Maxwell's laws from physics which are used to model electromagnetism.

The study of multivariable calculus is a wonderful and thrilling experience. We have only touched on a few concepts from multivariable calculus here, as a brief introduction. Also, we have used *Mathematica* more extensively than at any other time in the course!

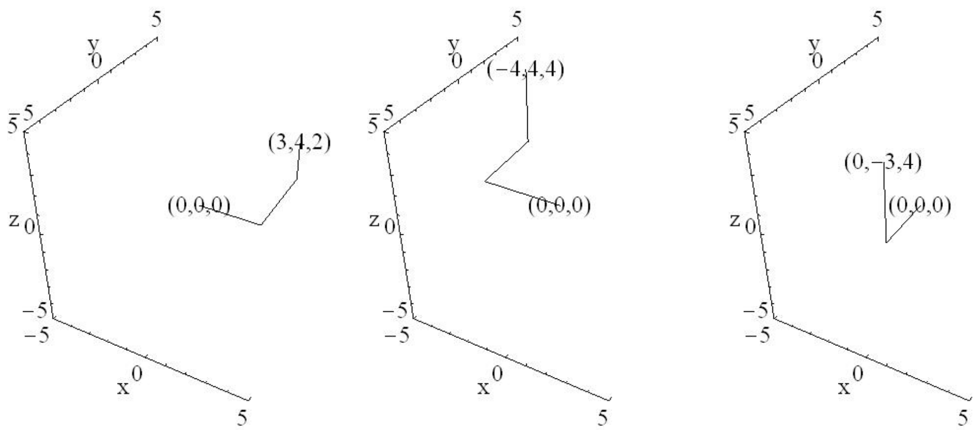
The concepts we focused on were
- $\mathbb{R}^3$, and a three dimensional coordinate system,
- explicit representation of surfaces in $\mathbb{R}^3$, $z = f(x, y)$,
- traces in zx-plane and zy-plane (traces are families of curves),
- contour plots in xy-plane (lines in a contour plot connect points with same z value),
- parametric representations of space curves in $\mathbb{R}^3$,
- partial derivatives of explicit functions $z = f(x, y)$,
- short introduction to extrema in $\mathbb{R}^3$.

References

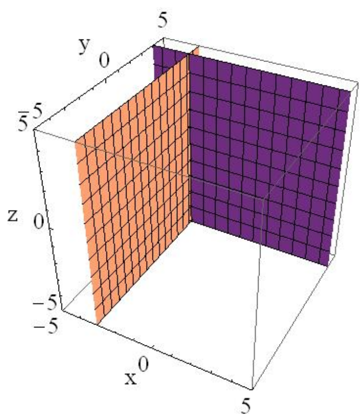
[1] J. Stewart, *Multivariable Calculus, 4th Ed.* (Brooks Cole Publishing, New York, 1999).

6 Solutions

Problem 1.



Problem 2.



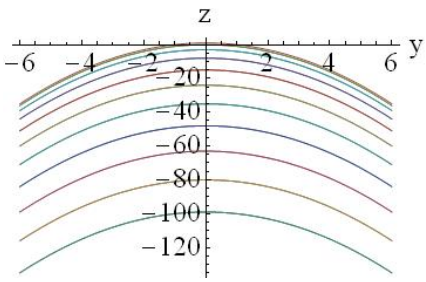
Problem 3. The trace in the zx-plane is given by the family of curves

$$z = 1 - x^2 - k^2.$$

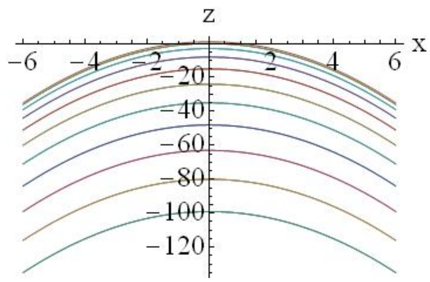
The trace in the zy-plane is given by the family of curves

$$z = 1 - k^2 - y^2.$$

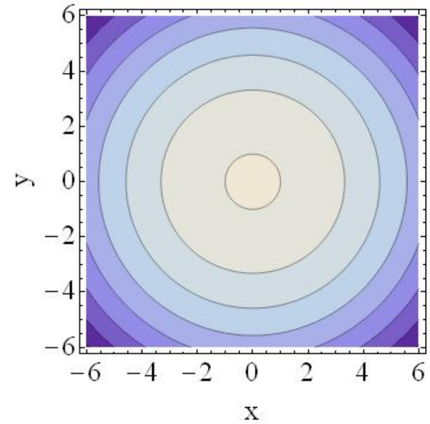
Sketches of these traces are given in Fig. 19. In the zy-plane we have a family of curves which are parabolas opening down, and in the zx-plane we have a family of curves which are also parabolas opening down. The contour plot shows circles, and since the middle is lighter we expect to find a maximum around the origin.




```
f[x_, y_] = 1 - x^2 - y^2
list = Table[f[k, y],
   {k, -10, 10}]
Plot[Evaluate[list],
   {y, -6, 6}]
```




```
f[x_, y_] = 1 - x^2 - y^2
list = Table[f[x, k],
    {k, -10, 10}]
Plot[Evaluate[list],
    {x, -6, 6}]
```




```
f[x_, y_] = 1 - x^2 - y^2
ContourPlot[f[x, y],
  {x, -6, 6}, {y, -6, 6}]
```

FIGURE 19: THE TRACES IN THE zx-PLANE (CENTER) AND THE zy-PLANE (LEFT) AND THE CONTOUR PLOT (RIGHT) FOR THE SURFACE $z = 1 - x^2 - y^2$.

Problem 4. The trace in the zx-plane is given by the family of curves

$$x^2 + k^2 + z^2 = 1.$$

The trace in the zy-plane is given by the family of curves

$$k^2 + y^2 + z^2 = 1.$$

Sketches of these traces are given in Fig. 20. In the zy-plane we have a family of curves which are circles, and in the zx-plane we have a family of curves which are ellipses.

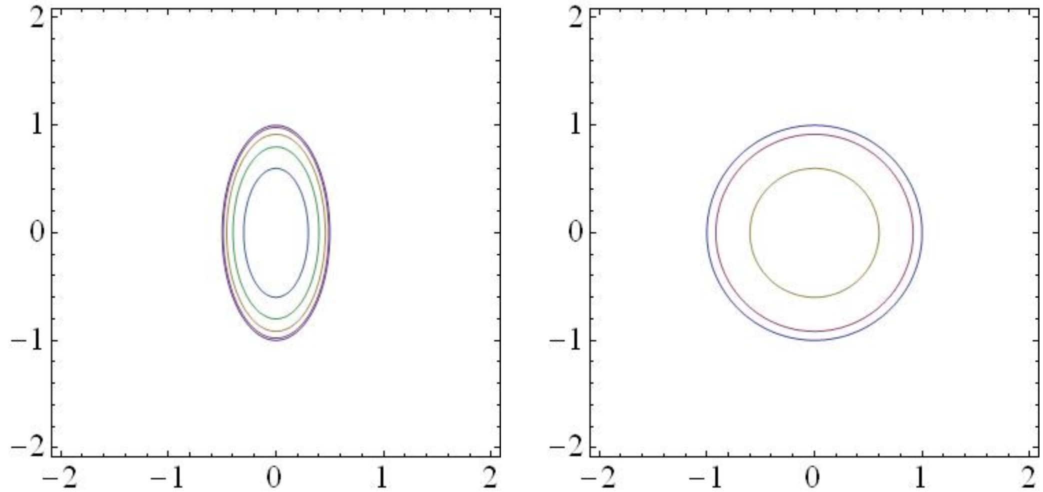
Problem 5. The trace in the zx-plane is given by the family of curves

$$z = \cos(xk)e^{(-x^2-k^2)/10}.$$

The trace in the zy-plane is given by the family of curves

$$z = \cos(ky)e^{(-k^2-y^2)/10}.$$

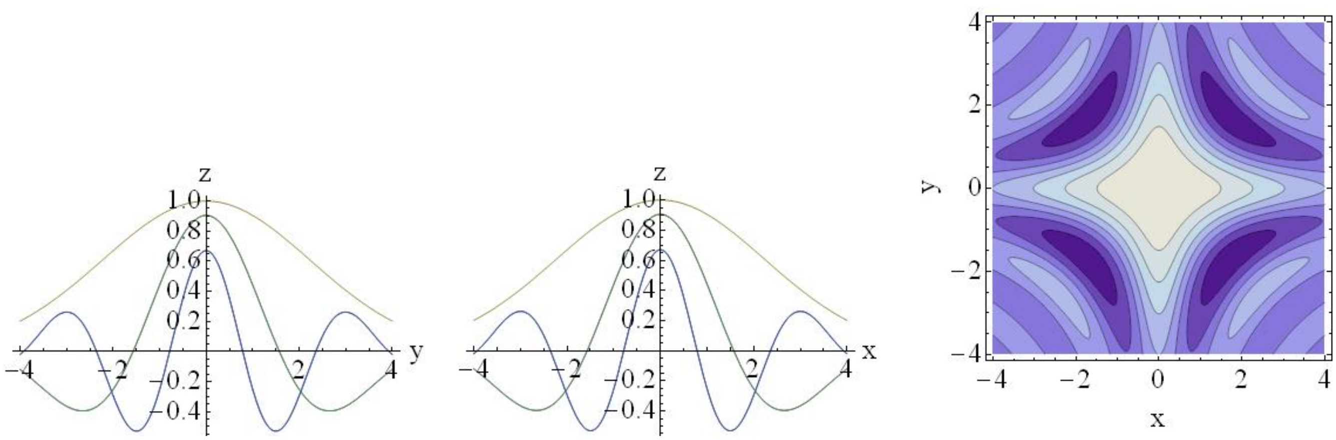
Sketches of these traces are given in Fig. 21. In the zy-plane we have a family of curves which oscillate, and also decay; and in the zx-plane we have a family of curves which also oscillates and decays.



```
f[x_, y_, z_] = 1 - x^2 - 4y^2 - z^2
list = Table[f[k,y,z]==0,{k,0,2,0.2}]
ImplicitPlot[Evaluate[list],
  {y,-2,2},{z,-2,2},
  PlotPoints -> 40]
```

```
list = Table[f[x, k, z] == 0, {k, 0, 2, 0.2}]
ImplicitPlot[Evaluate[list], {x, -2, 2},
  {z, -2, 2},
  PlotPoints -> 40]
```

Figure 20: The traces in the zx-plane (right) and the zy-plane (left) for the surface $x^2 + 4y^2 + z^2 = 1$.



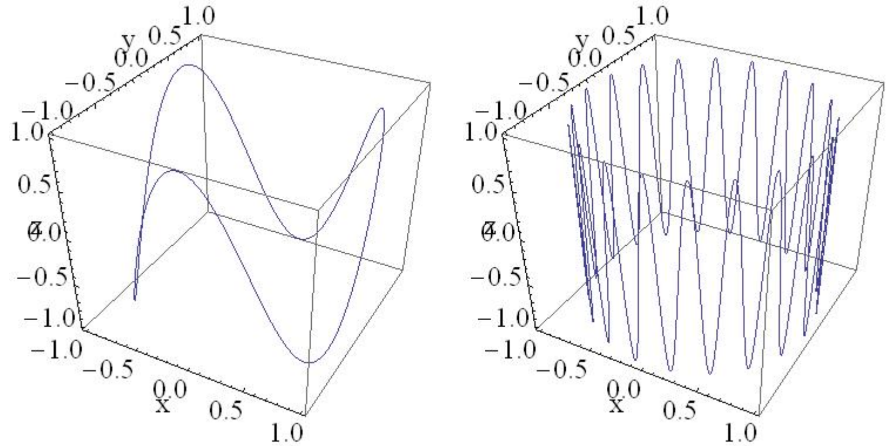

```
f[x_, y_] =
   Cos[y x]*Exp[(-y^2-x^2)/10]
list = Table[f[k, y],
   {k, -2, 2}]
Plot[Evaluate[list],
   {y, -4, 4}]
```

```
f[x_, y_] =
   Cos[y x]*Exp[(-y^2-x^2)/10]
list = Table[f[x, k],
    {k, -2, 2}]
Plot[Evaluate[list],
    {x, -4, 4}]
```

```
f[x_, y_] =
    Cos[y x]*Exp[(-y^2-x^2)/10]
ContourPlot[f[x, y],
  {x, -6, 6}, {y, -6, 6}]
```

Figure 21: The traces in the zx-plane (center) and the zy-plane (left) and the contour plot (right) for the surface $z = \cos(xy)e^{(-x^2-y^2)/10}$.

Problem 6. The function is sketched for $a = 3$ and $a = 20$ in Fig. 22. The function has "peaks", and it seems like it will have a peaks. Also, the function is cyclic in that it looks like it will repeat if we take a larger set of t. Finally, for a large the function gets "choppy" when *Mathematica* plots it. This is because the number of points that *Mathematica* is sampling at is too small to reproduce the structure of the function accurately. The space curve can be made smoother by increasing the number of plot points used.

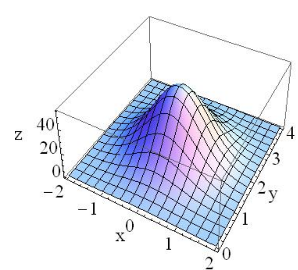



```
a = 3
ParametricPlot3D[
  {Cos[t], Sin[t], Sin[a t]},
  {t, 0, 2Pi}, AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z"}]
```

```
a = 3
ParametricPlot3D[
  {Cos[t], Sin[t], Sin[a t]},
  {t, 0, 2Pi}, AspectRatio -> 1,
  AxesLabel -> {"x", "y", "z"},
  PlotPoints -> 1000]
```

FIGURE 22: THE SPACE CURVES $x = \cos t, y = \sin t, z = \sin at, \quad 0 \leq t \leq 2\pi$, WITH $a = 3$ AND $a = 20$.

Problem 11. First, we can plot the function. The sketch can be seen in Fig. 23. Now we know we are looking for a




```
f[x_, y_] = Exp[4y - x^2 - y^2]
Plot3D[f[x, y], {x,-2,2}, {y,0,4}, AspectRatio->1,
       AxesLabel -> {"x","y","z"}]
```

FIGURE 23: THE SURFACE $z = f(x, y) = e^{4y-x^2-y^2}$.

maximum. We can find the point where the maximum occurs with the help of *Mathematica*:

```
eq1 = D[f[x, y], x] == 0
eq2 = D[f[x, y], y] == 0
NSolve[{eq1, eq2}, {x, y}]
```

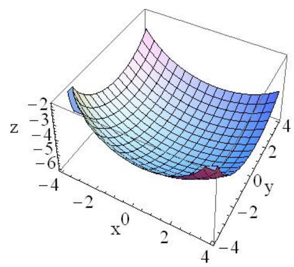
We could also do this by hand to find

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{4y-x^2-y^2}(-2x) = 0 \\
\frac{\partial f}{\partial y} &= e^{4y-x^2-y^2}(4-2y) = 0
\end{aligned}$$

The only solution to these equations is $(x, y) = (0, 2)$, and this point seems reasonable based on our sketch. Therefore, the function has a local minimum at $(x, y) = (0, 2)$ of $f(0, 2) = e^4$.

Problem 12.

First, we can plot the function. The sketch can be seen in Fig. 24.




```
f[x_, y_] = -Sqrt[36 - x^2 - y^2 + x + 2y]
Plot3D[f[x, y], {x,-4,4}, {y,-4,4}]
```

FIGURE 24: THE SURFACE $z = f(x, y) = -\sqrt{36 - x^2 - y^2 + x + 2y}$.

Now we know we are looking for a minimum. We can find the point where the minimum occurs by calculating partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{1-2x}{2\sqrt{36 - x^2 - y^2 + x + 2y}} = 0 \\
\frac{\partial f}{\partial y} &= -\frac{2-2y}{2\sqrt{36 - x^2 - y^2 + x + 2y}} = 0
\end{aligned}$$

or

$$1 - 2x = 0, \qquad 2 - 2y = 0.$$

The only solution to these equations is $(x, y) = (1/2, 1)$, and this point seems reasonable based on our sketch. Therefore, the function has a local minimum at $(x, y) = (1/2, 1)$ of $f(1/2, 1) = -\sqrt{149}/2$.