Questions

Example Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their domains if $f(x) = 1 - x^3$ and $g(x) = 1/x$.

Example Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their domains if $f(x) = \sqrt{x-1}$ and $g(x) = x^2$.

Example Find $f \circ g \circ h$ where $f(x) = 1/x$, $g(x) = x^3$ and $h(x) = x - 1$.

Example Find $f \circ g \circ h$ where $f(x) = 2/(1+x)$, $g(x) = \cos x$ and $h(x) = \sqrt{x+3}$.

Example The *Heaviside function* H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

a) Sketch the graph of the Heaviside function.
b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ seconds and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
c) Sketch a graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

Example The Heaviside function defined above can also be used to define the *ramp function* $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

a) Sketch the graph of the ramp function $y = tH(t)$.
b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ seconds and the voltage is gradually increased to 120 volts over a 60 second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
c) Sketch a graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

Solutions

Example Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their domains if $f(x) = 1 - x^3$ and $g(x) = 1/x$.

a) First, get the composition:

$$\begin{aligned} (f \circ g)(x) &= f\big(g(x)\big) \\ &= f(1/x) \\ &= 1 - \left(\frac{1}{x}\right)^3 \\ &= 1 - \frac{1}{x^3} \end{aligned}$$

Now we can get the domain for the resulting function. As long as we do not have division by zero, this function is defined, which means $x \neq 0$. The domain of $g(x)$ is also all x not equal to zero.

The domain of $f \circ g$ is therefore $x \in (-\infty, 0) \cup (0, \infty)$.

b) First, get the composition:

$$\begin{aligned} (g \circ f)(x) &= g\big(f(x)\big) \\ &= g(1 - x^3) \\ &= \frac{1}{1 - x^3} \end{aligned}$$

This is undefined if the denominator equals zero, which happens if $1 - x^3 = 0$. The only real valued solution to this equation is $x = 1$. The function $f(x)$ has domain all real numbers, so the domain of $g \circ f$ is therefore $x \in (-\infty, 1) \cup (1, \infty)$.

c) First, get the composition:

$$\begin{aligned} (f \circ f)(x) &= f\big(f(x)\big) \\ &= f(1 - x^3) \\ &= 1 - (1 - x^3)^3 \\ &= 1 - \big(1(1)^3(-x^3)^0 + 3(1)^2(-x^3)^1 + 3(1)^1(-x^3)^2 + 1(1)^0(-x^3)^3\big) \text{ (using Pascal's Triangle)} \\ &= 1 - \big(1 - 3x^3 + 3x^6 - 1x^9\big) \text{ (using Pascal's Triangle to expand power)} \\ &= 3x^3 - 3x^6 + 1x^9 \end{aligned}$$

This is a polynomial, therefore, the domain is $x \in \mathbb{R}$. Since f also has domain $x \in \mathbb{R}$, the domain of $f \circ f$ is $x \in \mathbb{R}$.

d) First, get the composition:

$$\begin{aligned} (g \circ g)(x) &= g\big(g(x)\big) \\ &= g(1/x) \\ &= (1/x)^{-1} \\ &= x \end{aligned}$$

This is simply a polynomial, so has a domain of $x \in \mathbb{R}$. However, g has a domain $\{x \in \mathbb{R} | x \neq 0\}$, so $g \circ g$ has domain $\{x \in \mathbb{R} | x \neq 0\}$.

Example Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their domains if $f(x) = \sqrt{x - 1}$ and $g(x) = x^2$.

a) First, get the composition:

$$\begin{aligned} (f \circ g)(x) &= f\big(g(x)\big) \\ &= f(x^2) \\ &= \sqrt{x^2 - 1} \end{aligned}$$

Now we can get the domain for the resulting function. The square root is defined on $\mathbb{R}$ for an argument which is greater than or equal to zero. This means we must have

$$\begin{aligned} x^2 - 1 &> 0 \\ x^2 &> 1 \end{aligned}$$

This condition holds if $x > 1$ or $x < -1$. Also, the domain of g is $x \in \mathbb{R}$. The domain of $f \circ g$ is therefore $x \in (-\infty, -1] \cup [1, \infty)$.

b) First, get the composition:

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) \\ &= g(\sqrt{x-1}) \\ &= (\sqrt{x-1})^2 \\ &= x - 1\end{aligned}$$

This is simply a polynomial, and so it has domain $x \in \mathbb{R}$. However, f has domain $x \geq 1$, so the domain of $g \circ f$ is $x \geq 1$.

c) First, get the composition:

$$\begin{aligned}(f \circ f)(x) &= f\big(f(x)\big) \\ &= f(\sqrt{x-1}) \\ &= \sqrt{\sqrt{x-1} - 1} \qquad (1)\end{aligned}$$

We cannot simplify this expression any further.

Now we can get the domain. The square root is defined on $\mathbb{R}$ for an argument which is greater than or equal to zero. Therefore, we must have that $x - 1 \geq 0$ for the square root in the inner square root in Eq. (1) to be defined. So we know that $x \geq 1$. We must also have the quantity inside the outer square root of Eq. (1) be greater than zero:

$$\begin{aligned}\sqrt{x-1} - 1 &\geq 0 \\ \sqrt{x-1} &\geq 1 \\ x - 1 &\geq 1 \\ x &\geq 2\end{aligned}$$

The fact that $x \geq 2$ contains the requirement that $x \geq 1$, so the domain for $f \circ f$ is $x \in [2, \infty)$.

d) First, get the composition:

$$\begin{aligned}(g \circ g)(x) &= g\big(g(x)\big) \\ &= g(x^2) \\ &= (x^2)^2 \\ &= x^4\end{aligned}$$

This is simply a polynomial, and so $g \circ g$ has a domain of $x \in \mathbb{R}$.

Example Find $f \circ g \circ h$ where $f(x) = 1/x$, $g(x) = x^3$ and $h(x) = x - 1$.

First, get the composition

$$(f \circ g \circ h)(x) = f\Big(g\big(h(x)\big)\Big)$$

$$
\begin{aligned}
&= f\Big(g\big(x^2+2\big)\Big) \\
&= f\big((x^2+2)^3\big) \\
&= \frac{1}{(x^2+2)^3}
\end{aligned}
$$

This is an algebraic function. The domain might possibly be constrained if there is division by zero. But $x^2 + 2 \neq 0$ for $x \in \mathbb{R}$, so the domain is $x \in \mathbb{R}$. The domains of h and g are $x \in \mathbb{R}$, so the domain of $f \circ g \circ h$ is $x \in \mathbb{R}$.

Example Find $f \circ g \circ h$ where $f(x) = 2/(1+x)$, $g(x) = \cos x$ and $h(x) = \sqrt{x+3}$.

First, get the composition

$$
\begin{aligned}
(f \circ g \circ h)(x) &= f\Big(g\big(h(x)\big)\Big) \\
&= f\Big(g\big(\sqrt{x+3}\big)\Big) \\
&= f\big(\cos(\sqrt{x+3})\big) \\
&= \frac{2}{1+\cos(\sqrt{x+3})}
\end{aligned}
$$

For the domain, we must have $x + 3 \geq 0$ for the square root to be defined, so $x \geq -3$. Also, we must exclude any values of x for which the denominator becomes zero. The cosine is -1 at angles of $(2n+1)\pi$, n an integer, and this will give a zero denominator and so must be excluded.

$$
\begin{aligned}
\sqrt{x+3} &= (2n+1)\pi \\
x+3 &= (2n+1)^2\pi^2 \\
x &= (2n+1)^2\pi^2 - 3
\end{aligned}
$$

The domain is therefore $x \geq -3$ where $x \neq (2n+1)^2\pi^2 - 3$ where n is an integer.

Example The *Heaviside function* H is defined by

$$
H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}
$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

a) Sketch the graph of the Heaviside function.
b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ seconds and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
c) Sketch a graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

The Heaviside function looks like the following

From Fig. 2, we can get the analytic equations for the voltages. For the case on the right (part b), we have simply changed the vertical scale on the Heaviside function shown in Fig. 1, so we have $V(t) = 120H(t)$.

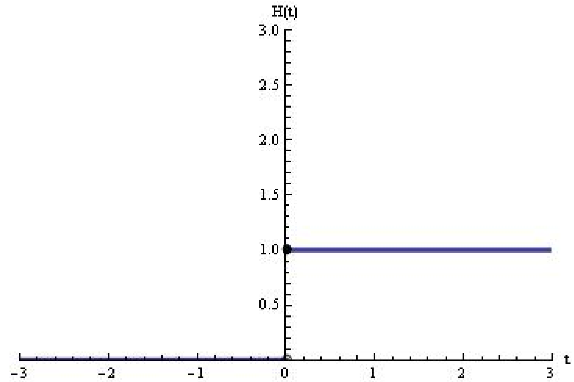

Figure 1: The graph of the Heaviside function.

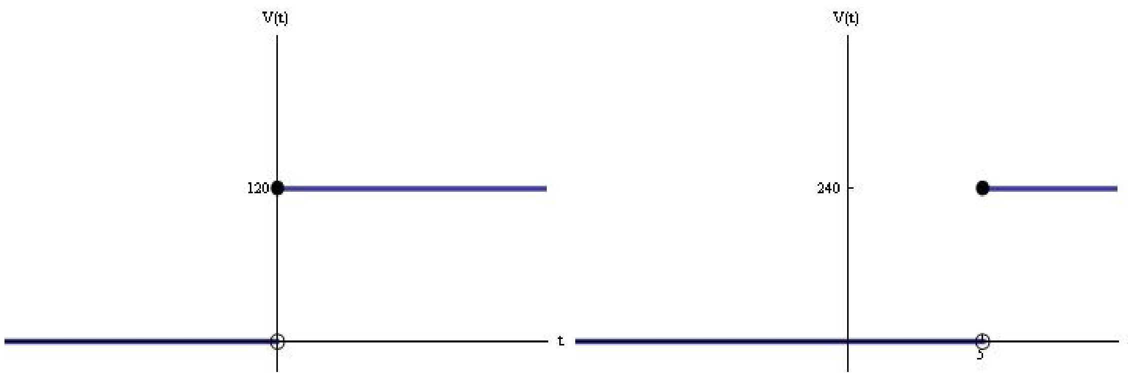

Figure 2: The graph of the Voltage functions for the two cases described in the question, parts b and c.

For the case on the left (part c), we have changed the vertical scale. This means we should write $V(t) = 240H(t)$. However, this function still has the voltage switching on at $t = 0$, not $t = 5$ as we as supposed to have!

Turning the voltage on at $t = 5$ means we have shifted the graph to the right by a constant 5. This means we should change t to $t - 5$. Therefore, we finally arrive at the analytic equation for the voltage, which is $V(t) = 240H(t - 5)$.

Example The Heaviside function defined above can also be used to define the *ramp function* $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.

a) Sketch the graph of the ramp function $y = tH(t)$.
b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ seconds and the voltage is gradually increased to 120 volts over a 60 second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
c) Sketch a graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

The ramp function which is described via the Heaviside function can be rewritten as a piecewise defined function explicitly by removing our Heaviside function. This may make it easier to see what we are graphing!

$$y(t) = tH(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$

Note: The Heaviside function is sometimes called a *step function* or even a *switching function*. You can see why it would be called a switching function, as it acts to *switch on* the linear part! From the piecewise definition, we can construct the graph.

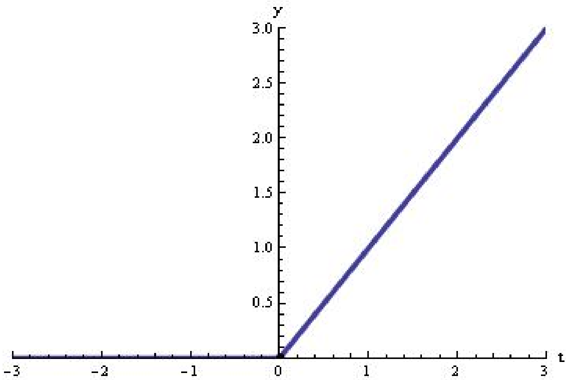

Figure 3: The graph of the ramp function.

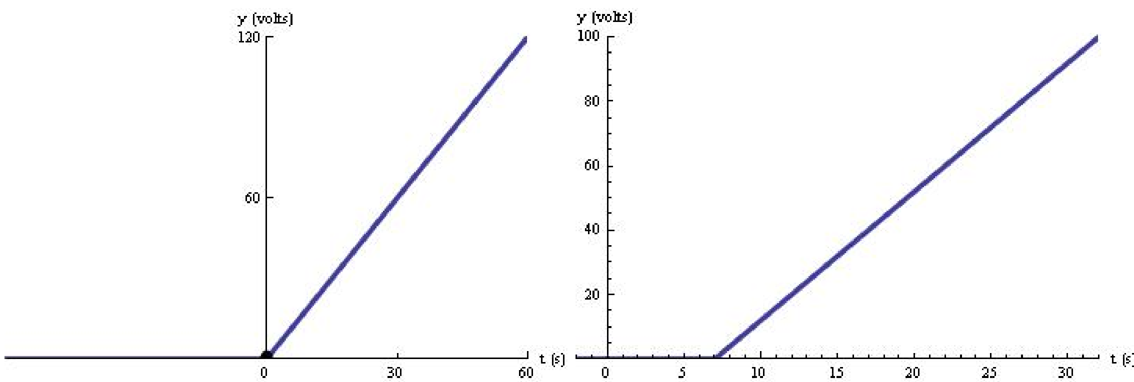

Figure 4: The graph of the Voltage functions for the two cases described in the question, parts b and c. We are assuming once the voltage is switched on that it increases linearly to the values that are given.

Now, all we have to do is get the analytic equations for the voltages. For the case on the left (part b), we see that the function looks like

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } t \geq 0 \end{cases} = 2t \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} = 2tH(t)$$

For the graph on the right (part c), let's proceed by comparing to the ramp function $y = ctH(t)$. First, the slope of the line must be c, and from the graph we can find the slope to be $c = 100/25 = 4$. Then, we have to deal with the shift to the right. A shift to the right by 7 units means we replace the t by $t - 7$. Therefore, the equation of the voltage for the case on the right is given by $y = 4(t - 7)H(t - 7)$.