Questions

Example Evaluate the limit and justify each step by indicating the appropriate limit law

$$\lim_{x\to -1} \frac{x-2}{x^2+4x-3}.$$

Example The *signum* or sign function , denoted by sgn, is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

a) Sketch the graph of this function.

b) Find each of the following limits or explain why it does not exist.

i) $\lim_{x\to 0^+} \operatorname{sgn}(x)$ ii) $\lim_{x\to 0^-} \operatorname{sgn}(x)$ iii) $\lim_{x\to 0} \operatorname{sgn}(x)$ iv) $\lim_{x\to 0} |\operatorname{sgn}(x)|$

Example If the symbol $[\![\cdot]\!]$ denotes the greatest integer function defined as $[\![x]\!] =$ the largest integer that is less than or equal to x, evaluate

i) $\lim_{x\to -2^+} [\![x]\!]$ ii) $\lim_{x\to -2} [\![x]\!]$ iii) $\lim_{x\to -2.4} [\![x]\!]$

b) If n is an integer, evaluate

i) $\lim_{x\to n^-} [\![x]\!]$ ii) $\lim_{x\to n^+} [\![x]\!]$

c) For what values of a does $\lim_{x\to a} [\![x]\!]$ exist?

Solutions

Example Evaluate the limit and justify each step by indicating the appropriate limit law

$$\lim_{x\to -1} \frac{x-2}{x^2+4x-3}.$$

$$\begin{aligned}
\lim_{x\to -1} \frac{x-2}{x^2+4x-3} &= \frac{\lim\limits_{x\to -1}(x-2)}{\lim\limits_{x\to -1}(x^2+4x-3)} && \text{law 5; as long as } \lim_{x\to -1}(x^2+4x-3) \neq 0 \\
&= \frac{\lim\limits_{x\to -1} x - \lim\limits_{x\to -1} 2}{\lim\limits_{x\to -1} x^2 + \lim\limits_{x\to -1} 4x - \lim\limits_{x\to -1} 3} && \text{law 1; law 2} \\
&= \frac{\lim\limits_{x\to -1} x - \lim\limits_{x\to -1} 2}{\left[\lim\limits_{x\to -1} x\right]^2 + 4\lim\limits_{x\to -1} x - \lim\limits_{x\to -1} 3} && \text{law 3; law 6} \\
&= \frac{(-1)-2}{(-1)^2+4(-1)-3} && \text{law 7; law 8} \\
&= \frac{-3}{-6} = \frac{1}{2}
\end{aligned}$$

Example The *signum* or sign function , denoted by sgn, is defined by

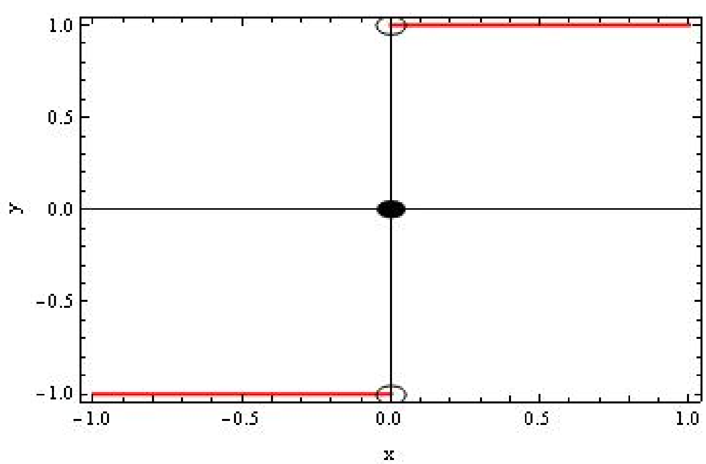
$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

a) Sketch the graph of this function.

b) Find each of the following limits or explain why it does not exist.

i) $\lim\limits_{x\to 0^+} \text{sgn}(x)$ ii) $\lim\limits_{x\to 0^-} \text{sgn}(x)$ iii) $\lim\limits_{x\to 0} \text{sgn}(x)$ iv) $\lim\limits_{x\to 0} |\text{sgn}(x)|$

Here is a sketch of the sgn function:

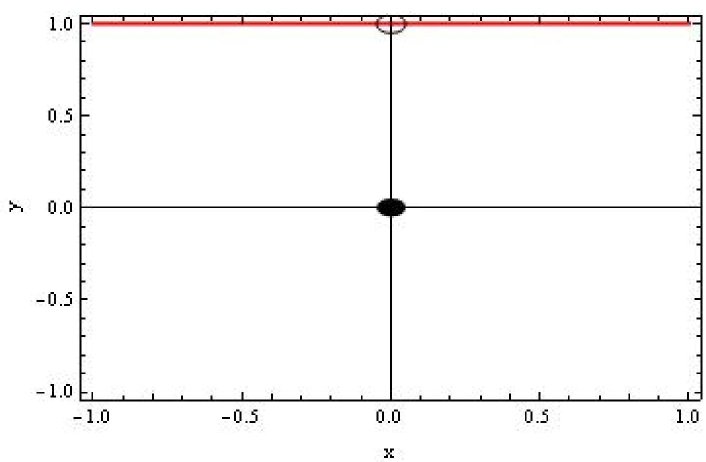


$$\begin{aligned}
\lim_{x\to 0^+} \text{sgn}(x) &= \lim_{x\to 0^+}(1) \quad \text{since } \text{sgn}(x) = 1 \text{ if } x > 0 \\
&= 1 \\
\lim_{x\to 0^-} \text{sgn}(x) &= \lim_{x\to 0^-}(-1) \quad \text{since } \text{sgn}(x) = -1 \text{ if } x < 0 \\
&= -1
\end{aligned}$$

Therefore, $\lim_{x\to 0} \text{sgn}(x)$ does not exist since the right hand limit does not equal the left hand limit.

Let's write down the functional definition of $|\text{sgn}(x)|$ and sketch a graph to help us find those limits:

$$\begin{aligned}
|\text{sgn}(x)| &= \begin{cases} \text{sgn}(x) & \text{if } \text{sgn}(x) \geq 0 \\ -\text{sgn}(x) & \text{if } \text{sgn}(x) < 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases} \\
&= \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}
\end{aligned}$$



$$\begin{aligned}
\lim_{x\to 0^+} |\text{sgn}(x)| &= \lim_{x\to 0^+}(1) \quad \text{since } |\text{sgn}(x)| = 1 \text{ if } x > 0 \\
&= 1 \\
\lim_{x\to 0^-} |\text{sgn}(x)| &= \lim_{x\to 0^-}(1) \quad \text{since } |\text{sgn}(x)| = 1 \text{ if } x < 0 \\
&= 1
\end{aligned}$$

Since the left hand limit equals the right hand limit, we have $\lim_{x\to 0} |\text{sgn}(x)| = 1$.

Example If the symbol $[\![\,\cdot\,]\!]$ denotes the greatest integer function defined as $[\![x]\!]=$ the largest integer that is less than or equal to x, evaluate

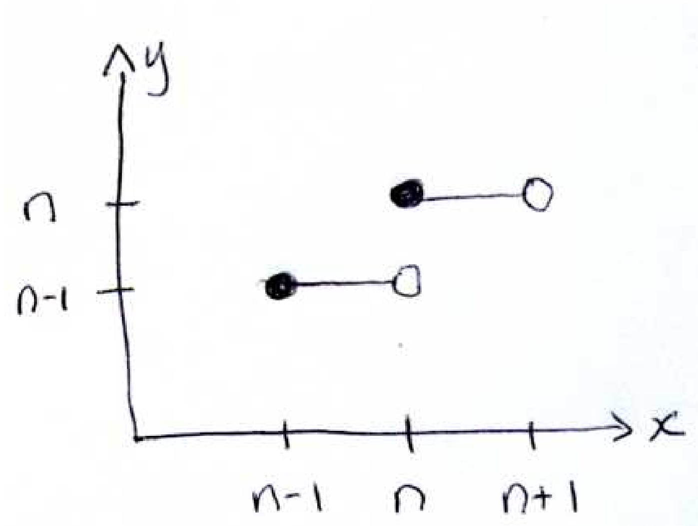
i) $\lim_{x\to -2^+} [\![x]\!]$ ii) $\lim_{x\to -2} [\![x]\!]$ iii) $\lim_{x\to -2.4} [\![x]\!]$

b) If n is an integer, evaluate

i) $\lim_{x\to n^-} [\![x]\!]$ ii) $\lim_{x\to n^+} [\![x]\!]$

c) For what values of a does $\lim_{x\to a} [\![x]\!]$ exist?

The greatest integer function is piecewise defined and changes definitions at integer values of x. The sketch below shows the greatest integer function in the region of the integer $x = n$. We can use this to help us answer questions regarding the limit.



$$
\begin{aligned}
\lim_{x\to -2^+} [\![x]\!] &= -2 \quad \text{(think of } n=-2 \text{ in the above sketch)} \\
\lim_{x\to -2^-} [\![x]\!] &= -3 \quad \text{(think of } n=-2 \text{ in the above sketch)} \\
\lim_{x\to -2} [\![x]\!] & \quad \text{does not exist since left hand limit does not equal right hand limit} \\
\lim_{x\to -2.4^-} [\![x]\!] &= -3 \quad \text{(think of } n=-2 \text{ in the above sketch)} \\
\lim_{x\to n^+} [\![x]\!] &= n \\
\lim_{x\to n^-} [\![x]\!] &= n-1
\end{aligned}
$$

The values of a for which $\lim_{x\to a} [\![x]\!]$ exists is all a which are not an integer.