Questions

Example A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part of an hour), up to a daily maximum of \$10.

a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.

b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

Example Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

$$g(x) = \frac{x+1}{2x^2-1}, \quad a = 4.$$

Example Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

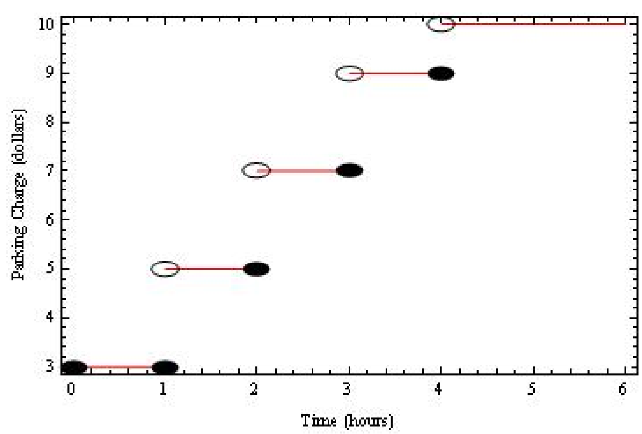
Solutions

Example A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part of an hour), up to a daily maximum of \$10.

a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.

b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

Here is a sketch of the parking lot problem:



There are discontinuities at $t = 1, 2, 3, 4$ hours. The discontinuities means the parker will pay a significant amount extra if they park for only a few minutes past an hour (or two hours, or three hours, or four hours). Get your shopping done a few minutes before the hour is up, or you shall pay dearly! If you are going to take slightly longer than an hour, you may as well go get a coke and relax in the food court for a half hour, since you are going to pay for parking for the full hour to come anyway. You will be paying for parking time which you do not use.

I would say that discontinuities always mean you are going to pay more than you would if the pay function was continuous. Remember the phone companies who would charge long distance by rounding up to the next minute? Same thing as here. You are paying for services which you do not use. The phone companies who round to the second are still doing the same thing, but you are paying for a fraction of a second of unused phone time instead of a fraction of a minute.

Taxes, user fees, and socialism are all involved in this idea. The extreme case is I pay for a service *I never use.* Personally, I think that sometimes this is a good thing (public transit, health care, education, etc).

Example Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

$$g(x) = \frac{x+1}{2x^2-1}, \quad a = 4.$$

Our function is $g(x) = \dfrac{x+1}{2x^2-1}$, and we want to investigate continuity at $a = 4$. First, we must evaluate the left and right hand limit.

$$\begin{aligned}
\lim_{x\to a^+} g(x) &= \lim_{x\to 4^+} \frac{x+1}{2x^2-1} \\
&= \frac{4+1}{2(4)^2-1} \quad \text{evaluate by direct substitution} \\
&= \frac{5}{31} \\
\lim_{x\to a^-} g(x) &= \lim_{x\to 4^-} \frac{x+1}{2x^2-1} \\
&= \frac{4+1}{2(4)^2-1} \quad \text{evaluate by direct substitution} \\
&= \frac{5}{31} \\
g(a) &= g(4) = \frac{4+1}{2(4)^2-1} \\
&= \frac{5}{31}
\end{aligned}$$

Since $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^-} g(x) = 5/31$, we can say that $\lim_{x\to 4} g(x) = 5/31$. Since $\lim_{x\to 4} g(x) = g(4) = 5/31$, we can say $g(x)$ is continuous at $x = 4$.

Example Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

If a function g is continuous at $x = a$, then

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = g(a).$$

Our function $g(x)$ is piecewise defined. For $x < 4$, it is the polynomial $x^2 - c^2$, so it is continuous (polynomials are continuous). For $x > 4$ it is also a polynomial, so it will also be continuous in this region. The only point we don't know if the function $g(x)$ is continuous is at $x = 4$, not surprisingly the point where the definition changes.

We must choose c to make the function continuous at $x = 4$. We do this by by imposing that the following limits be equal:

$$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^-} g(x).$$

Insert the proper definitions for $g(x)$:

$$\lim_{x\to 4^+} (cx + 20) = \lim_{x\to 4^-} (x^2 - c^2)$$

Evaluate by direct substitution:

$$c(4) + 20 = (4)^2 - c^2$$

A little algebraic rearranging gives us the following quadratic in c:

$$c^2 + 4c + 4 = 0$$

So if c satisfies this quadratic, then the left and right hand limits will be equal. The equality with $g(a)$ that is required for continuity follows automatically in this case. All that is left to do is solve the quadratic for c:

$$c(4) + 20 = (4)^2 + c^2 \quad\longrightarrow\quad (c+2)^2 = 0 \quad\longrightarrow\quad c = -2$$

So if $c = -2$, the function $g(x)$ will be continuous for $x \in \mathbb{R}$.