Questions

Example Make a careful sketch of the graph of $f(x) = \sin x$ and below it sketch the graph of $f'(x)$. Try to guess the formula of $f'(x)$ from its graph.

Example Make a careful sketch of the graph of $f(x) = \ln x$ and below it sketch the graph of $f'(x)$. Try to guess the formula of $f'(x)$ from its graph.

Example Prove each of the following:
a) The derivative of an even function is an odd function.
b) The derivative of an odd function is an even function.

Example Find the derivative by finding the first few derivative and observing the pattern that occurs. $D^{103} \cos 2x$.

Example A particle moves according to the law of motion $s(t) = t^3 - 12t^2 + 36t$, $t \geq 0$, where t is measured in seconds and s in meters.

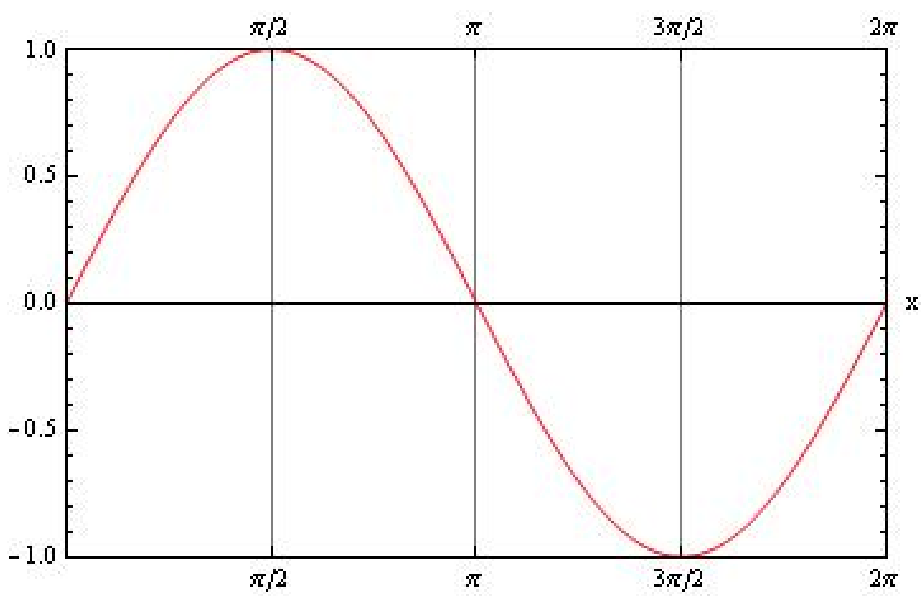
a) Find the acceleration at time t and after 3 s.
b) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 8$.
c) When is the particle speeding up? When is it slowing down?

Hint: Feel free to use *Mathematica* to evaluate and simplify the limits in this question, although you can do them by hand as well.

Solutions

Example Make a careful sketch of the graph of $f(x) = \sin x$ and below it sketch the graph of $f'(x)$. Try to guess the formula of $f'(x)$ from its graph.

1) This is neat! I've sketched the sine function, $y = f(x) = \sin x$. The key idea we use over and over again here is that the value of the derivative at $x = a$ is equal to the slope of the tangent line to the curve at $x = a$.



At the points where the sine function is ± 1 it has a horizontal tangent. This means that the derivative $f'(x)$ must be zero at those points. So $f'(\pi/2) = f'(3\pi/2) = 0$.

The sine function has a minimum slope at $x = \pi$, and if you look closely (or create a new graph by zooming in) you can guess that the value of that minimum slope is -1. This means that $f'(\pi) = -1$.

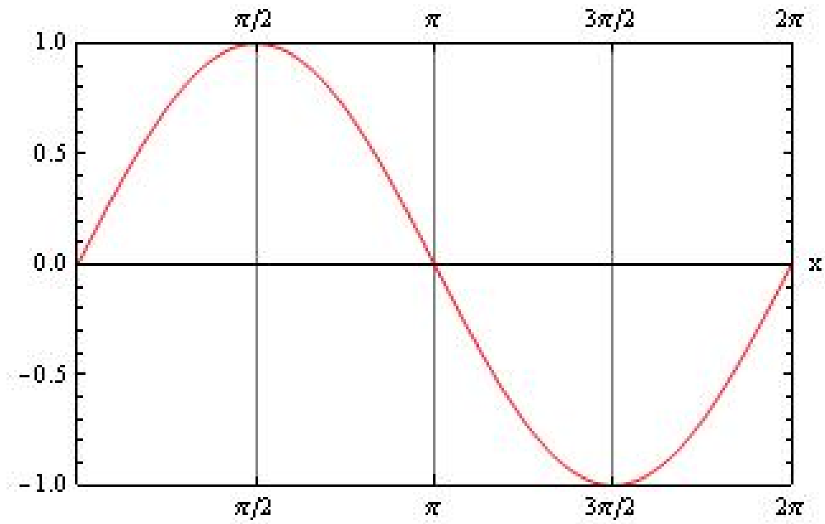
The sine function has a maximum slope at $x = 0$ or 2π, and if you look closely (or create a new graph by zooming in) you can guess that the value of that minimum slope is $+1$. This means that $f'(0) = f'(2\pi) = 1$.

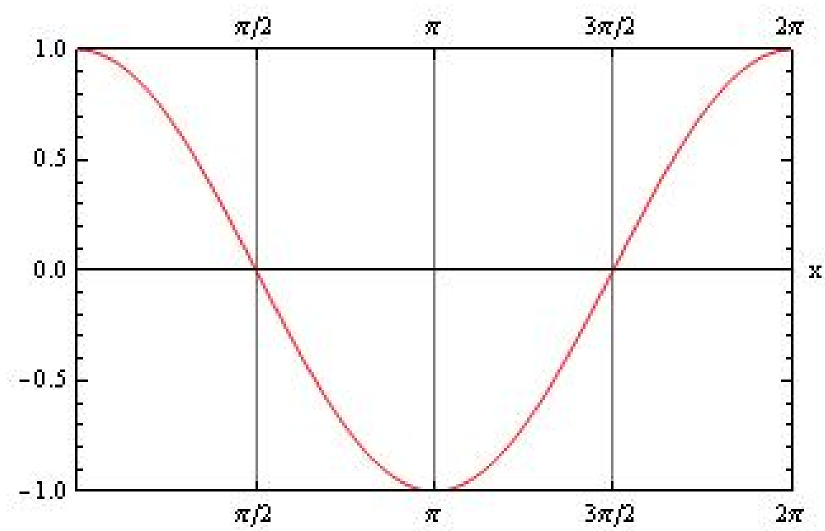
For $0 < x < \pi/2$, the sine function is increasing. This means its derivative will be positive in this region.

For $\pi/2 < x < 3\pi/2$, the sine function is decreasing. This means its derivative will be negative in this region.

For $3\pi/2 < x < 2\pi$, the sine function is increasing. This means its derivative will be positive in this region.

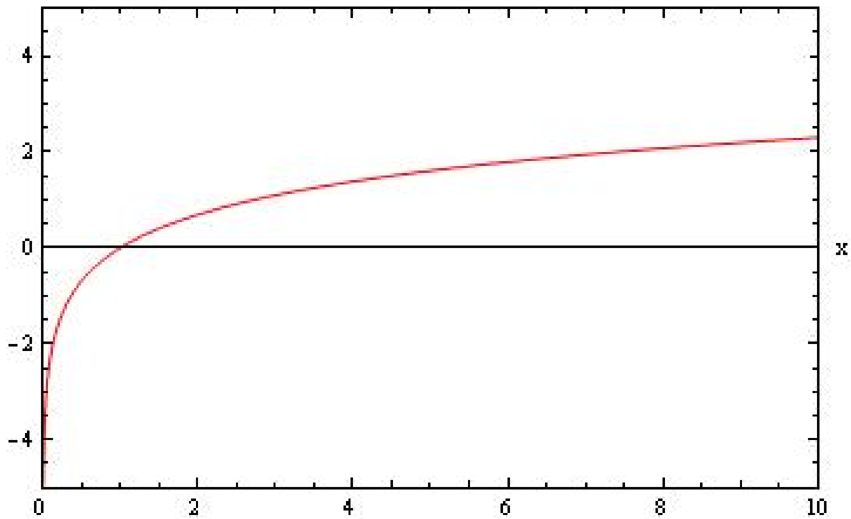
And that says in words every thought and relation that we used to construct the sketch of the derivative of the sine function, which is included below. The derivative sure looks like the *cosine* function!





Example Make a careful sketch of the graph of $f(x) = \ln x$ and below it sketch the graph of $f'(x)$. Try to guess the formula of $f'(x)$ from its graph.

2) Again, the key idea we use here is that the value of the derivative at $x = a$ is equal to the slope of the tangent line top the curve at $x = a$. First, let's sketch the logarithmic function $y = f(x) = \ln x$.



This one isn't as nice as the sine function we just finished, since there are fewer points to help guide us. We could zoom in on a bunch of points, work out the slope (since the graph will appear as a straight line when we zoom in) and then plot those values as points on the derivative curve. Joining the points would give us a graph of the derivative $f'(x)$.

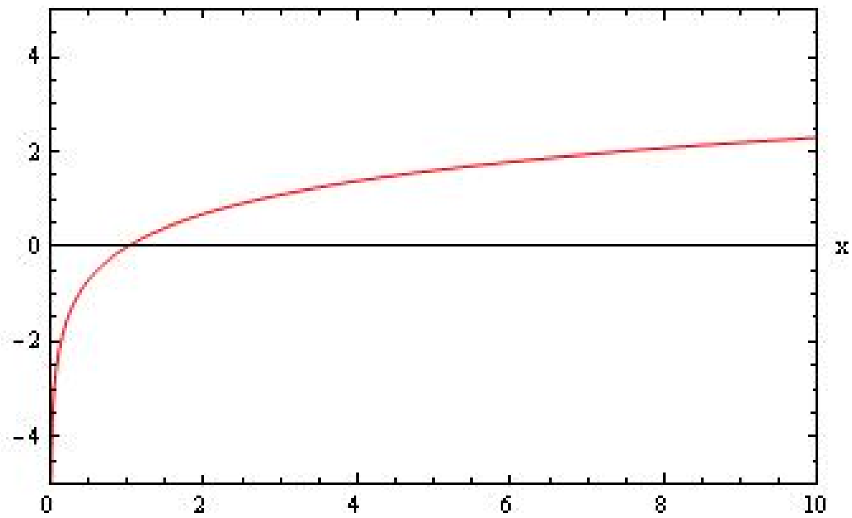
Instead, let's do what the problem suggests, and look at the graph in broader scope.

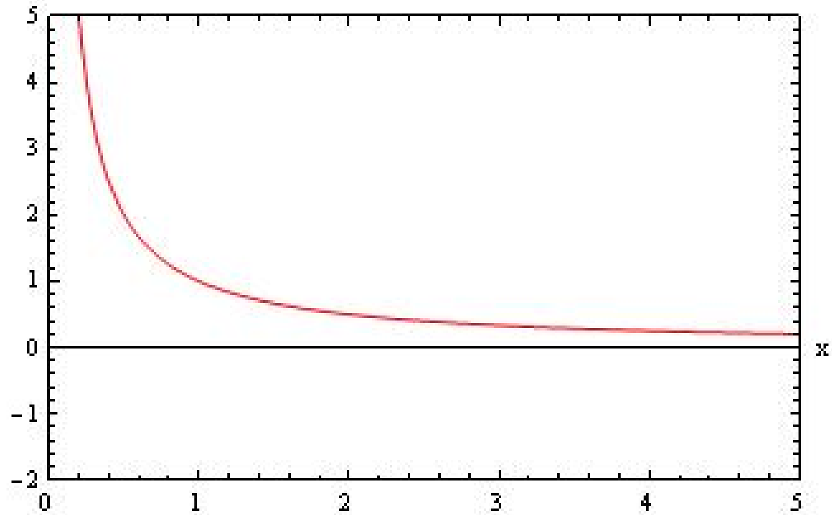
First, the logarithmic function is always increasing, so the derivative function will always be positive.

Second, as the logarithmic function approaches zero from the right, it grows infinitely large negative. This means our derivative function should "blow up" at $x = 0$, and since it has to be positive, it must approach positive infinity. So we know that $\lim_{x \to 0^+} f'(x) = \infty$.

The last thing we can say from looking at the graph of the logarithmic function is that the slope of tangent to the curve is decreasing. This means $f'(x)$ is a decreasing function of x.

I've included all these details in the sketch below. It seems to me that we may have $f'(x) = \dfrac{1}{x}$.





Example Prove each of the following:

a) The derivative of an even function is an odd function.

b) The derivative of an odd function is an even function.

a) Here, we need to work from the definition of derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If the function $f(x)$ is even, then $f(-x) = f(x)$. To show that the derivative is going to be odd, we need to show $f'(-x) = -f'(x)$.

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} \quad \text{since } f \text{ is even} \end{aligned}$$

Now we can't jump to our conclusion too quickly! This doesn't quite look like the definition of derivative we know and love. We really want $f(x+h)$ in there. So let's make a substitution $h = -\widetilde{h}$, and see what happens. As $h \to 0$, so will

$\widetilde{h} \to 0$.

$$\begin{aligned} f'(-x) &= \lim_{\widetilde{h}\to 0} \frac{f(x+\widetilde{h})-f(x)}{-\widetilde{h}} \\ &= -\lim_{\widetilde{h}\to 0} \frac{f(x+\widetilde{h})-f(x)}{\widetilde{h}} \\ &= -f'(x) \end{aligned}$$

So we have proven that if f is even, then f' is odd.

We will see another way of proving this statement once we have learned the chain rule for derivatives.

b) Here, we need to work from the definition of derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

If the function $f(x)$ is odd, then $f(-x) = -f(x)$. To show that the derivative is going to be even, we need to show $f'(-x) = f'(x)$.

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0} \frac{-f(x-h)+f(x)}{h} \quad \text{since } f \text{ is odd} \\ &= -\lim_{h\to 0} \frac{f(x-h)-f(x)}{h} \quad \text{since } f \text{ is odd} \end{aligned}$$

Just like above, we need to make a substitution before we are done. Let's make the substitution $h = -\widetilde{h}$, and see what happens. As $h \to 0$, so will $\widetilde{h} \to 0$.

$$\begin{aligned} f'(-x) &= -\lim_{\widetilde{h}\to 0} \frac{f(x+\widetilde{h})-f(x)}{-\widetilde{h}} \\ &= +\lim_{\widetilde{h}\to 0} \frac{f(x+\widetilde{h})-f(x)}{\widetilde{h}} \\ &= +f'(x) \end{aligned}$$

So we have proven that if f is odd, then f' is even.

Example Find the derivative by finding the first few derivative and observing the pattern that occurs. $D^{103}\cos 2x$.

$$\begin{aligned} D^0\cos 2x &= +2^0\cos 2x \text{ (I added this just help with the modulus below)} \\ D^1\cos 2x &= -2^1\sin 2x \\ D^2\cos 2x &= -2^2\cos 2x \\ D^3\cos 2x &= +2^3\sin 2x \\ D^4\cos 2x &= +2^4\cos 2x \end{aligned}$$

So the functional part repeats in steps of four. Since 103 modulus 4 is 3, we know the derivative we seek will have a functional part of $+\sin 2x$. For the power of 2, the pattern is obvious, and we have $D^{103}\cos 2x = +2^{103}\sin 2x$.

Example A particle moves according to the law of motion $s = f(t) = t^3 - 12t^2 + 36t$, $t \geq 0$, where t is measured in seconds and s in meters.

a) Find the acceleration at time t and after 3 s.
b) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 8$.
c) When is the particle speeding up? When is it slowing down?

Hint: Feel free to use *Mathematica* to evaluate and simplify the limits in this question, although you can do them by hand as well.

The velocity is the derivative of the position, and the acceleration is related to the position by the second derivative. Therefore,

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s[t_] = t^3 - 12 t^2 + 36 t
v[t_] = Limit[(s[t + h] - s[t])/h, h -> 0]
a[t_] = Limit[(v[t + h] - v[t])/h, h -> 0]
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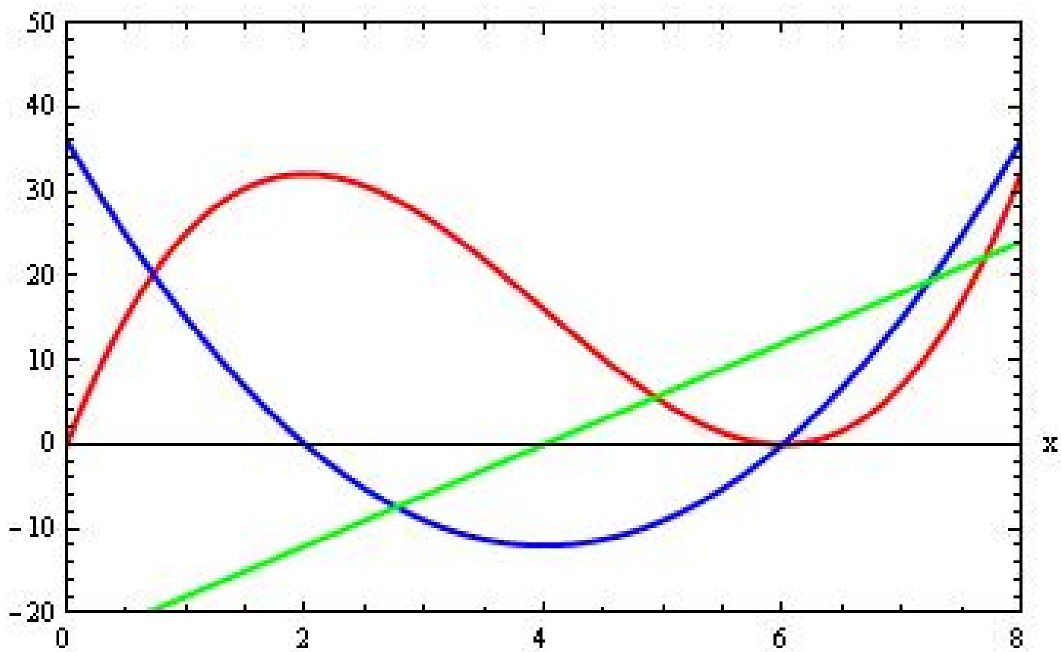
This is the acceleration at any time t. After 3 s, the acceleration is $a(3) = 6(3) - 24 = -6$ m/s^2.

Here is a sketch, where the linear (green) line is the acceleration, the parabolic (blue) line is the velocity, and the cubic (red) line is the position.



The particle is speeding up when the velocity and acceleration have the same sign. From the graph, the particle is speeding up for $2 < t < 4$ and $6 < t < 8$, and the particle is slowing down for $0 < t < 2$ and $4 < t < 6$.