

Questions

Example Find the linearization $L(x)$ of the function $f(x) = (x)^{1/3}$ at $a = -8$.

Example Find the linear approximation of the function $g(x) = (1+x)^{1/3}$ at $a = 0$ and use it to approximate the numbers $(0.95)^{1/3}$ and $(1.1)^{1/3}$. Illustrate by graphing $g(x)$ and the tangent line.

Example Use differentials (or, equivalently, a linear approximation) to estimate the number $\ln 1.07$.

Example On page 431 of *Physics: Calculus*, 2d ed, by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L , the author obtains the equation $a_T = -g \sin \theta$ for the tangential acceleration of the bob of the pendulum. He then says “for small angles, the value of θ in radians is very nearly the value of $\sin \theta$; they differ by less than 2% out to about 20° .”

Verify the linear approximation at 0 for the sine function, $\sin x \sim x$. Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht’s statement by converting from radians to degrees.

Solutions

Example Find the linearization $L(x)$ of the function $f(x) = (x)^{1/3}$ at $a = -8$.

The linearization is given by

$$L(x) = f(a) + f'(a)(x - a)$$

which approximates the function $f(x)$ near $x = a$.

We need the function and derivative evaluated at $a = -8$:

$$\begin{aligned} f(x) &= (x)^{1/3} \\ f(-8) &= (-8)^{1/3} \\ &= -2 \text{ (the only real valued result)} \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{1}{3}(x)^{-2/3} \\ f'(-8) &= \frac{1}{3}(-8)^{-2/3} \\ &= \frac{1}{3}(-2)^{-2} \\ &= \frac{1}{3} \cdot \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= f(-8) + f'(-8)(x - (-8)) \\ &= -2 + \left(\frac{1}{12}\right)(x + 8) \end{aligned}$$

Example Find the linear approximation of the function $g(x) = (1+x)^{1/3}$ at $a = 0$ and use it to approximate the numbers $(0.95)^{1/3}$ and $(1.1)^{1/3}$. Illustrate by graphing $g(x)$ and the tangent line.

The linearization is given by

$$L(x) = g(a) + g'(a)(x - a)$$

which approximates the function $g(x)$ near $x = a$.

We need the function and derivative evaluated at $a = 0$:

$$\begin{aligned}g(x) &= (1+x)^{1/3} \\g(0) &= (1+0)^{1/3} \\&= 1 \\g'(x) &= \frac{1}{3}(1+x)^{-2/3}(+1) \text{ (chain rule)} \\&= \frac{1}{3(1+x)^{2/3}} \\g'(0) &= \frac{1}{3(1+0)^{2/3}} \\&= \frac{1}{3} \\L(x) &= g(0) + g'(0)(x - 0) \\&= g(0) + g'(0)(x - 0) \\&= 1 + \left(\frac{1}{3}\right)(x) \\&= \frac{x}{3} + 1\end{aligned}$$

We can now use the linearization to approximate the two numbers.

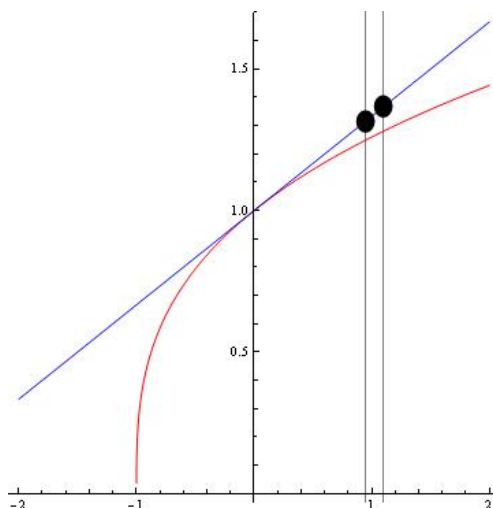
$$\begin{aligned}(0.95)^{1/3} &= (1 + (-0.05))^{1/3} \\&= g(-0.05) \\&\sim L(-0.05) \\&= \frac{-0.05}{3} + 1 \\&= 0.98333\end{aligned}$$

So we have $(0.95)^{1/3} \sim 0.98333$.

$$\begin{aligned}(1.1)^{1/3} &= (1 + (0.1))^{1/3} \\&= g(0.1) \\&\sim L(0.1) \\&= \frac{0.1}{3} + 1 \\&= 1.03333\end{aligned}$$

So we have $(1.1)^{1/3} \sim 1.03333$.

Here is a sketch of the situation:



The blue line is the tangent line $L(x)$, the red line is the function $g(x)$, and the dots are where we evaluated to estimate the two numbers. We are evaluating along the tangent line rather than along the function $g(x)$. We do this because it is easier to compute a numerical value along the tangent line than to compute a cube root directly. The further we move away from the center of the linearization $a = 0$, the worse our approximation generally becomes. We see that the approximation near $x = 0.1$ is worse than the approximation near $x = -0.5$, since the tangent line is further away from the curve $g(x)$ at $x = 0.1$.

Example Use differentials (or, equivalently, a linear approximation) to estimate the number $\ln 1.07$.

Using differentials would simply change the notation slightly. Let's use the linearization notation.

The linearization is given by $L(x) = f(a) + f'(a)(x - a)$ which approximates the function $f(x)$ near $x = a$.

We need to identify what $f(x)$ should be, and what our center should be. We are guided in our choice, because we want to be able to evaluate $f(a)$ and $f'(a)$ easily.

An example of a poor choice would be $f(x) = \ln x$ and $a = 1.07$, since we would need to evaluate $f(a) = \ln 1.07$, which is what we are trying to avoid!

Instead, think of an easy number to evaluate the logarithm at. For logarithms, the easiest number is 1, since $\ln 1 = 0$. Then, we will want x to be a *small distance away from this number*. This is the essence of the linearization procedure. So the function we want to linearize is $f(x) = \ln(1 + x)$, and we will choose a center of $a = 0$.

We need the function and derivative evaluated at $a = 0$:

$$f(x) = \ln(1+x)$$

$$f(0) = \ln 1$$

$$= 0$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(0) = \frac{1}{1+0}$$

$$= 1$$

$$L(x) = f(a) + f'(a)(x - a)$$

$$= f(0) + f'(0)(x - (0))$$

$$= 0 + 1(x)$$

$$= x$$

We can now use the linearization to approximate the number.

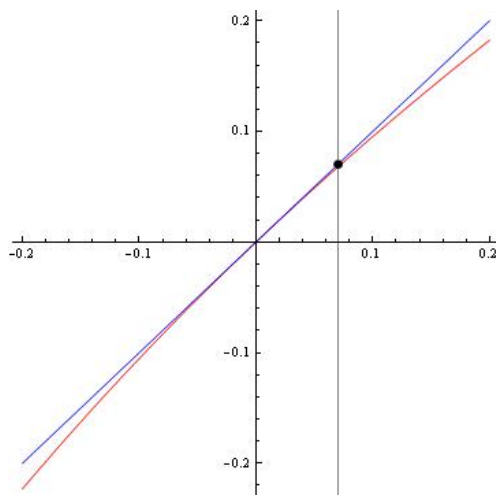
$$\ln 1.07 = \ln(1 + 0.07)$$

$$= f(0.07)$$

$$\sim L(0.07)$$

$$= 0.07$$

Here is a sketch of the situation.



The blue line is the tangent line $L(x)$, the red line is the function $f(x)$, and the dot is where we evaluated to estimate $\ln 1.07$.

We could have chosen a different function and a , and still got the result. Consider the function $f(x) = \ln(1 - 7x)$, and we will choose a center of $a = 0$.

We need the function and derivative evaluated at $a = 0$:

$$f(x) = \ln(1 - 7x)$$

$$f(0) = \ln 1$$

$$= 0$$

$$f'(x) = \frac{1}{1 - 7x} (-7)$$

$$f'(0) = (-7) \frac{1}{1 + 0}$$

$$= -7$$

$$L(x) = f(a) + f'(a)(x - a)$$

$$= f(0) + f'(0)(x - (0))$$

$$= 0 - 7(x)$$

$$= -7x$$

We can now use this linearization to approximate the number.

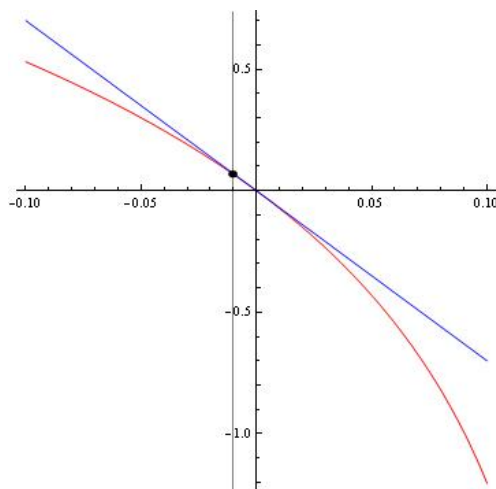
$$\ln 1.07 = \ln(1 - 7(-0.01))$$

$$= f(-0.01)$$

$$\sim L(-0.01)$$

$$= -7(-0.01) = 0.07$$

Here is a sketch of this situation.



The blue line is the tangent line $L(x)$, the red line is the function $f(x)$, and the dot is where we evaluated to estimate $\ln 1.07$.

Example On page 431 of *Physics: Calculus*, 2d ed, by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L , the author obtains the equation $a_T = -g \sin \theta$ for the tangential acceleration of the bob of the pendulum. He then says “for small angles, the value of θ in radians is very nearly the value of $\sin \theta$; they differ by less than 2% out to about 20° .”

Verify the linear approximation at 0 for the sine function, $\sin x \sim x$. Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

Consider the function $f(x) = \sin x$, and we will choose a center of $a = 0$.

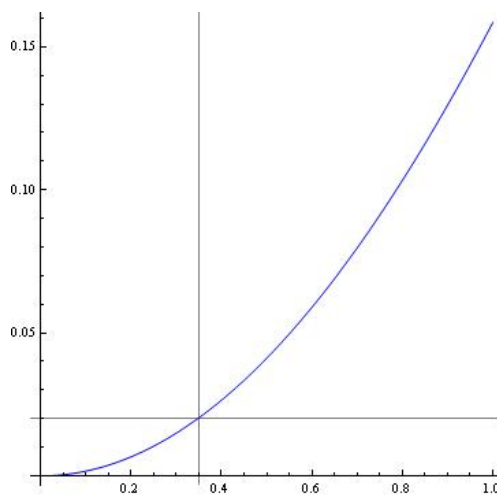
We need the function and derivative evaluated at $a = 0$:

$$\begin{aligned} f(x) &= \sin x \\ f(0) &= \sin 0 \\ &= 0 \\ f'(x) &= \cos x \\ f'(0) &= \cos 0 \\ &= 1 \\ L(x) &= f(a) + f'(a)(x - a) \\ &= f(0) + f'(0)(x - (0)) \\ &= 0 + 1(x) \\ &= x \end{aligned}$$

So we have shown that $\sin x \sim x$ if $x \sim 0$.

If we want the approximation to be good to 2%, then the difference between the two should be less than 2%. The easiest way to find when this happens is to plot $y = |(\sin x - x)/x|$.

Here is a sketch of this situation.



From the sketch, it looks like they differ by about 2% when $x = 0.35$ radians $= 360(0.35)/(2\pi) = 20.0535^\circ$.