Example (3.11.8) Find the linearization $L(x)$ of the function $f(x) = (x)^{1/3}$ at $a = -8$.

The linearization is given by

$$L(x) = f(a) + f'(a)(x - a)$$

which approximates the function $f(x)$ near $x = a$.

We need the function and derivative evaluated at $a = -8$:

$$\begin{aligned}
f(x) &= (x)^{1/3} \\
f(-8) &= (-8)^{1/3} \\
&= -2 \text{ (the only real valued result)} \\
f'(x) &= \frac{1}{3}(x)^{-2/3} \\
f'(-8) &= \frac{1}{3}(-8)^{-2/3} \\
&= \frac{1}{3}(-2)^{-2} \\
&= \frac{1}{3} \cdot \frac{1}{4} \\
&= \frac{1}{12} \\
L(x) &= f(a) + f'(a)(x - a) \\
&= f(-8) + f'(-8)(x - (-8)) \\
&= -2 + \left(\frac{1}{12}\right)(x + 8)
\end{aligned}$$

Example (3.11.10) Find the linear approximation of the function $g(x) = (1 + x)^{1/3}$ at $a = 0$ and use it to approximate the numbers $(0.95)^{1/3}$ and $(1.1)^{1/3}$. Illustrate by graphing $g(x)$ and the tangent line.

The linearization is given by

$$L(x) = g(a) + g'(a)(x - a)$$

which approximates the function $g(x)$ near $x = a$.

We need the function and derivative evaluated at $a = 0$:

$$\begin{aligned}
g(x) &= (1 + x)^{1/3} \\
g(0) &= (1 + 0)^{1/3} \\
&= 1 \\
g'(x) &= \frac{1}{3}(1 + x)^{-2/3}(+1) \text{ (chain rule)} \\
&= \frac{1}{3(1 + x)^{2/3}}
\end{aligned}$$

$$
\begin{aligned}
g'(0) &= \frac{1}{3(1+0)^{2/3}} \\
&= \frac{1}{3} \\
L(x) &= g(a) + g'(a)(x-a) \\
&= g(0) + g'(0)(x-0) \\
&= 1 + \left(\frac{1}{3}\right)(x) \\
&= \frac{x}{3} + 1
\end{aligned}
$$

We can now use the linearization to approximate the two numbers.

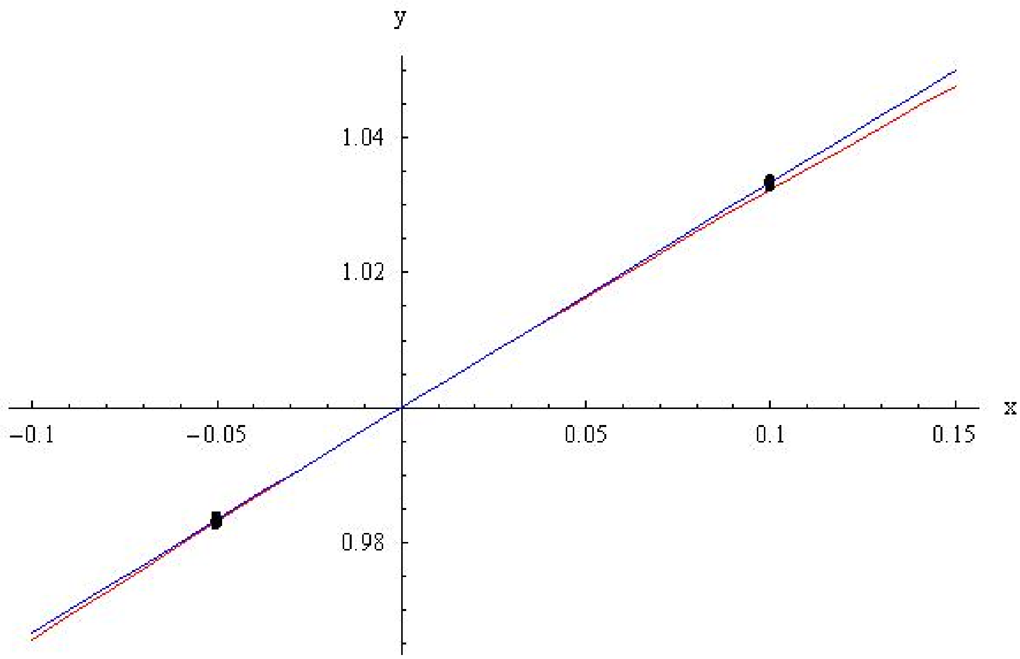
$$
\begin{aligned}
(0.95)^{1/3} &= (1+(-0.05))^{1/3} \\
&= g(-0.05) \\
&\sim L(-0.05) \\
&= \frac{-0.05}{3} + 1 \\
&= 0.98333
\end{aligned}
$$

So we have $(0.95)^{1/3} \sim 0.98333$.

$$
\begin{aligned}
(1.1)^{1/3} &= (1+(0.1))^{1/3} \\
&= g(0.1) \\
&\sim L(0.1) \\
&= \frac{0.1}{3} + 1 \\
&= 1.03333
\end{aligned}
$$

So we have $(1.1)^{1/3} \sim 1.03333$.

Here is a sketch of the situation:



The blue line is the tangent line $L(x)$, the red line is the function $g(x)$, and the dots are where we evaluated to estimate the two numbers. We are evaluating along the tangent line rather than along the function $g(x)$. We do this because it is easier to compute a numerical value along the tangent line than to compute a cube root directly. The further we move away from the center of the linearization $a = 0$, the worse our approximation generally becomes. We see that the approximation near $x = 0.1$ is worse than the approximation near $x = -0.5$, since the tangent line is further away from the curve $g(x)$ at $x = 0.1$.

Example (3.10.36) Use differentials (or, equivalently, a linear approximation) to estimate the number $\ln 1.07$.

Using differentials would simply change the notation slightly. Let's use the linearization notation.

The linearization is given by $L(x) = f(a) + f'(a)(x - a)$ which approximates the function $f(x)$ near $x = a$.

We need to identify what $f(x)$ should be, and what our center should be. We are guided in our choice, because we want to be able to evaluate $f(a)$ and $f'(a)$ easily.

An example of a poor choice would be $f(x) = \ln x$ and $a = 1.07$, since we would need to evaluate $f(a) = \ln 1.07$, which is what we are trying to avoid!

Instead, think of an easy number to evaluate the logarithm at. For logarithms, the easiest number is 1, since $\ln 1 = 0$. Then, we will want x to be a *small distance away from this number*. This is the essence of the linearization procedure. So the function we want to linearize is $f(x) = \ln(1 + x)$, and we will choose a center of $a = 0$.

We need the function and derivative evaluated at $a = 0$:

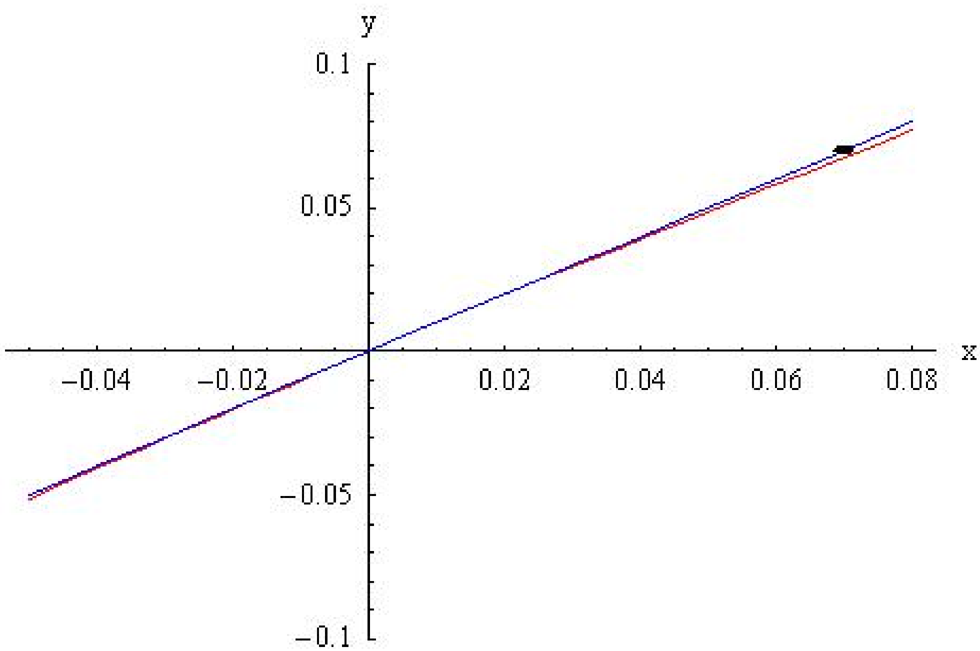
$$\begin{aligned} f(x) &= \ln(1 + x) \\ f(0) &= \ln 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} \\
f'(0) &= \frac{1}{1+0} \\
&= 1 \\
L(x) &= f(a) + f'(a)(x-a) \\
&= f(0) + f'(0)(x-(0)) \\
&= 0 + 1(x) \\
&= x
\end{aligned}$$

We can now use the linearization to approximate the number.

$$\begin{aligned}
\ln 1.07 &= \ln(1+0.07) \\
&= f(0.07) \\
&\sim L(0.07) \\
&= 0.07
\end{aligned}$$

Here is a sketch of the situation.



The blue line is the tangent line $L(x)$, the red line is the function $f(x)$, and the dot is where we evaluated to estimate $\ln 1.07$.

We could have chosen a different function and a, and still got the result. Consider the function $f(x) = \ln(1-7x)$, and we will choose a center of $a = 0$.

We need the function and derivative evaluated at $a = 0$:

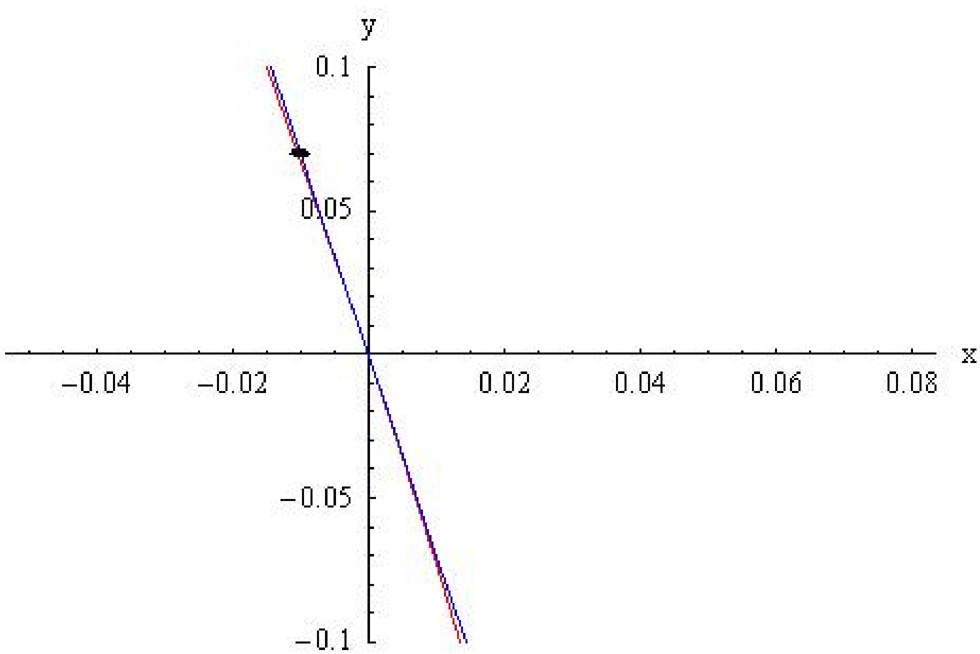
$$f(x) = \ln(1-7x)$$

$$
\begin{aligned}
f(0) &= \ln 1 \\
&= 0 \\
f'(x) &= \frac{1}{1-7x}(-7) \\
f'(0) &= (-7)\frac{1}{1+0} \\
&= -7 \\
L(x) &= f(a) + f'(a)(x-a) \\
&= f(0) + f'(0)(x-(0)) \\
&= 0 - 7(x) \\
&= -7x
\end{aligned}
$$

We can now use this linearization to approximate the number.

$$
\begin{aligned}
\ln 1.07 &= \ln(1-7(-0.01)) \\
&= f(-0.01) \\
&\sim L(-0.01) \\
&= -7(-0.01) = 0.07
\end{aligned}
$$

Here is a sketch of this situation.



The blue line is the tangent line $L(x)$, the red line is the function $f(x)$, and the dot is where we evaluated to estimate $\ln 1.07$.

Example (3.11.48) On page 431 of *Physics: Calculus*, 2d ed, by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L, the author obtains the equation $a_T = -g\sin\theta$ for the tangential acceleration of the bob of the pendulum. He then says "for small angles, the value of θ in radians is very nearly the value of $\sin\theta$; they differ by less than 2% out to about 20°."

Verify the linear approximation at 0 for the sine function, $\sin x \sim x$. Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

Consider the function $f(x) = \sin x$, and we will choose a center of $a = 0$.

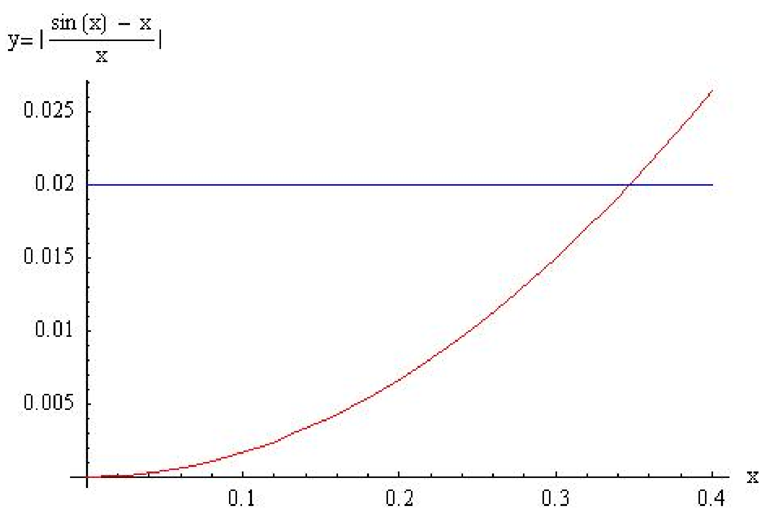
We need the function and derivative evaluated at $a = 0$:

$$\begin{aligned}
f(x) &= \sin x \\
f(0) &= \sin 0 \\
&= 0 \\
f'(x) &= \cos x \\
f'(0) &= \cos 0 \\
&= 1 \\
L(x) &= f(a) + f'(a)(x-a) \\
&= f(0) + f'(0)(x-(0)) \\
&= 0 + 1(x) \\
&= x
\end{aligned}$$

So we have shown that $\sin x \sim x$ if $x \sim 0$.

If we want the approximation to be good to 2%, then the difference between the two should be less than 2%. The easiest way to find when this happens is to plot $y = |(\sin x - x)/x|$.

Here is a sketch of this situation.



From the sketch, it looks like they differ by about 2% when $x = 0.35$ radians $= 360(0.35)/(2\pi) = 20.0535°$.