Questions

Example Find the derivative of the function $f(t) = (1 + \tan t)^{1/3}$.

Example Find the derivative of the function $y = (x^2 + 1)(x^2 + 2)^{1/3}$.

Example Find the derivative of the function $y = \sin(\tan \sqrt{\sin x})$.

Solutions

Example Find the derivative of the function $f(t) = (1 + \tan t)^{1/3}$.

$$\begin{aligned}
f(t) &= (1+\tan t)^{1/3} \\
\frac{d}{dt} f(t) &= \frac{d}{dt}[(1+\tan t)^{1/3}] \\
&= \frac{d}{dt}[u^{1/3}], \quad u = 1 + \tan t \\
&= \frac{d}{du}[u^{1/3}] \cdot \frac{du}{dt}, \text{ (by the chain rule)} \\
&= \left(\frac{1}{3}u^{-2/3}\right) \cdot \frac{d}{dt}[1 + \tan t] \\
&= \left(\frac{1}{3}(1+\tan t)^{-2/3}\right)(\sec^2 t), \text{ (back substitute)} \\
&= \frac{1}{3}\sec^2 t(1+\tan t)^{-2/3} \\
&= \frac{\sec^2 t}{3(1+\tan t)^{2/3}}
\end{aligned}$$

Example Find the derivative of the function $y = (x^2 + 1)(x^2 + 2)^{1/3}$.

$$\begin{aligned}
y &= (x^2+1)(x^2+2)^{1/3} \\
\frac{dy}{dt} &= \frac{d}{dt}[(x^2+1)(x^2+2)^{1/3}] \\
&= (x^2+1)\frac{d}{dx}[(x^2+2)^{1/3}] + (x^2+2)^{1/3}\frac{d}{dx}[(x^2+1)] \text{ (product rule)} \\
&= (x^2+1)\frac{d}{dx}[(u)^{1/3}] + (x^2+2)^{1/3}(2x), \quad u = x^2 + 2 \text{ (getting ready to use chain rule)} \\
&= (x^2+1)\frac{d}{du}[(u)^{1/3}] \cdot \frac{du}{dx} + 2x(x^2+2)^{1/3} \text{ (chain rule)} \\
&= (x^2+1)\left(\frac{1}{3}(u)^{-2/3}\right) \cdot (2x) + 2x(x^2+2)^{1/3} \\
&= (x^2+1)\left(\frac{1}{3}(x^2+2)^{-2/3}\right) \cdot (2x) + 2x(x^2+2)^{1/3} \text{ (back substitute)} \\
&= \frac{2x(x^2+1)}{3(x^2+2)^{2/3}} + 2x(x^2+2)^{1/3} \text{ (simplify)}
\end{aligned}$$

Example Find the derivative of the function $y = \sin(\tan\sqrt{\sin x})$.

We will need multiple applications of the chain rule to do this derivative. Let's do that first before we take any derivatives.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[\sin(\tan\sqrt{\sin x})] \\
&= \frac{d}{dx}[\sin u], && u = \tan\sqrt{\sin x} \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dx}, && \text{(chain rule)} \\
& && u = \tan v, v = \sqrt{\sin x} \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dx}, && \text{(chain rule a second time)} \\
& && u = \tan v, v = \sqrt{w}, w = \sin x \\
&= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}, && \text{(chain rule a third time)}
\end{aligned}$$

All this was just setting up the derivative in a manner that we could find it by using multiple applications of the chain rule. Now we can take the derivatives.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{du}[\sin u]\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx} \\
&= \frac{d}{du}[\sin u]\cdot\frac{d}{dv}[\tan v]\cdot\frac{d}{dw}[w^{1/2}]\cdot\frac{d}{dx}[\sin x] \\
&= (\cos u)\cdot(\sec^2 v)\cdot\left(\frac{1}{2}w^{-1/2}\right)\cdot(\cos x) \\
&= (\cos u)\cdot(\sec^2 v)\cdot\left(\frac{1}{2\sqrt{w}}\right)\cdot(\cos x) \\
&= (\cos(\tan\sqrt{\sin x}))\cdot(\sec^2\sqrt{\sin x})\cdot\left(\frac{1}{2\sqrt{\sin x}}\right)\cdot(\cos x) \\
&= \frac{\cos x\cos(\tan\sqrt{\sin x})\sec^2\sqrt{\sin x}}{2\sqrt{\sin x}}
\end{aligned}$$