Questions

Example If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$. What is the maximum height reached by the ball? What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

Example Find the average rate of change of the area of a circle with respect to its radius r as r changes from i) 2 to 3; ii) 2 to 2.5; iii) 2 to 2.1.

Find the instantaneous rate of change when $r = 2$. Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically how this is true by drawing a circle whose radius is increasing by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

Example If, in Example 4 in text, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentration of A and B have a common value [A] = [B] = a moles/L, then $[\mathrm{C}] = \dfrac{a^2kt}{akt+1}$ where k is a constant.

a)Find the rate of reaction at time t.

b) Show that if $x = [\mathrm{C}]$, then $\dfrac{dx}{dt} = k(a-x)^2$.

c) What happens to the concentration as $t \longrightarrow \infty$?

d) What happens to the rate of reaction as $t \longrightarrow \infty$?

e) What do the results of parts c) and d) mean in practical terms?

Solutions

Example If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$. What is the maximum height reached by the ball? What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

The maximum height reached by the ball will occur at the time when the ball has zero velocity.

The velocity is given by the derivative of the position function, $v = ds/dt$.

Therefore, we must solve the equation $ds/dt = 0$ for t, and the use this value of t to determine the maximum height the ball reaches.

$$\begin{aligned} s(t) &= 80t - 16t^2 \\ v(t) &= \frac{ds}{dt} \\ &= \frac{d}{dt}[80t - 16t^2] \\ &= 80 - 32t \end{aligned}$$

Now, we solve $v = 80 - 32t = 0$ for t, which yields $t = 80/32 = 5/2$ s. The maximum height of the ball occurs at $t = 5/2$ s.

The maximum height is $s(5/2) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 40(5) - 4(25) = 100$ ft.

To find the velocity when the ball is 96 ft above the ground on the way up we need to know the time when the ball is 96 feet above the ground. We find that by solving $s(t) = 96 = 80t - 16t^2$ for t, which can be done using the quadratic equation. We find the ball is 96 ft above the ground at 2 and 3 s. At 2 s it is on its way up, and at 3 s it is on its way down.

The velocities of the ball at these times are $v(2) = 80 - 32(2) = 16$ ft/s (ball at 96 feet on way up), and $v(3) = 80 - 32(3) = -16$ ft/s (ball at 96 feet on way down).

Example Find the average rate of change of the area of a circle with respect to its radius r as r changes from i) 2 to 3; ii) 2 to 2.5; iii) 2 to 2.1.

Find the instantaneous rate of change when $r = 2$. Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically how this is true by drawing a circle whose radius is increasing by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

The area of a circle is a function of the radius, $A(r) = \pi r^2$. The average rate of change of the area of a circle with respect to its radius is given by

$$\frac{\Delta A}{\Delta r} = \frac{\pi r_1^2 - \pi r_2^2}{r_1 - r_2}$$

where the radius is changing from r_1 to r_2.

i) $\dfrac{\Delta A}{\Delta r} = \dfrac{\pi(2)^2 - \pi(3)^2}{2 - 3} = 5\pi$ ii) $\dfrac{\Delta A}{\Delta r} = \dfrac{\pi(2)^2 - \pi(2.5)^2}{2 - 2.5} = 4.5\pi$ iii) $\dfrac{\Delta A}{\Delta r} = \dfrac{\pi(2)^2 - \pi(2.1)^2}{2 - 2.1} = 4.1\pi$

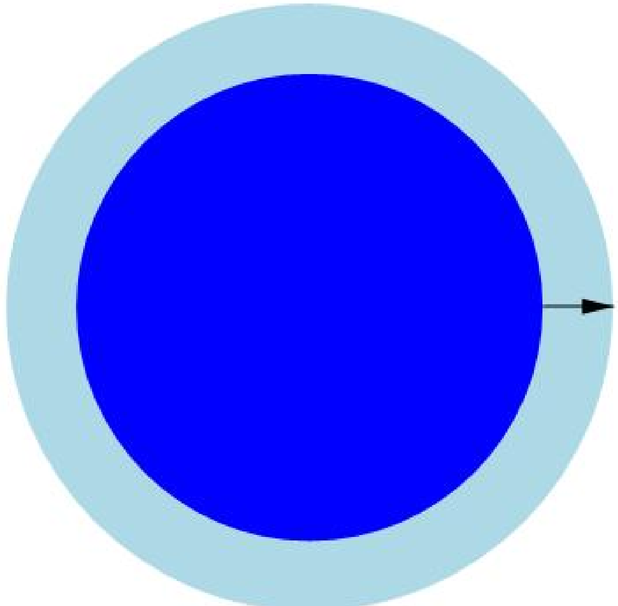
From these results, it looks like the instantaneous rate of change when $r = 2$ is 4π.

The instantaneous rate of change is given by the derivative. Therefore,

$$\frac{dA}{dr} = 2\pi r$$

which is the circumference of a circle of radius r.

Here is a picture of two circles, one with radius r (dark blue) and the other with radius $r + \Delta r$ (light blue).



The length of the arrow is Δr. The change in area ΔA between the two circles is the area contained in the light blue washer, which is

$$\begin{aligned} \Delta A &= \pi(r+\Delta r)^2 - \pi r^2 \\ &= \pi(r^2 + (\Delta r)^2 + 2r(\Delta r)) - \pi r^2 \\ &= \pi((\Delta r)^2 + 2r(\Delta r)) \end{aligned}$$

Now, we can make an approximation. If Δr is very small, then $(\Delta r)^2$ will be much much smaller than Δr, which we write as $(\Delta r)^2 << \Delta r$. This means we can neglect the term which has $(\Delta r)^2$.

$$\begin{aligned} \Delta A &= \pi((\Delta r)^2 + 2r(\Delta r)) \\ &\sim \pi(2r(\Delta r)) \\ \frac{\Delta A}{\Delta r} &\sim 2\pi r \text{ if } \Delta r \sim 0 \end{aligned}$$

We have shown geometrically why the instantaneous change in the area is equal to the circumference.

Example If, in Example 4 in text, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentration of A and B have a common value [A] = [B] = a moles/L, then $[\mathrm{C}] = \dfrac{a^2kt}{akt+1}$ where k is a constant.

a)Find the rate of reaction at time t.

b) Show that if $x = [\mathrm{C}]$, then $\dfrac{dx}{dt} = k(a-x)^2$.

c) What happens to the concentration as $t \longrightarrow \infty$?

d) What happens to the rate of reaction as $t \longrightarrow \infty$?

e) What do the results of parts c) and d) mean in practical terms?

The rate of reaction is give by $d[\mathrm{C}]/dt$, so

$$
\begin{aligned}
\frac{d[\mathrm{C}]}{dt} &= \frac{dx}{dt} \\
&= \frac{d}{dt}\left[\frac{a^2kt}{akt+1}\right] \\
&= \frac{(akt+1)\frac{d}{dt}[a^2kt]-(a^2kt)\frac{d}{dt}[akt+1]}{(akt+1)^2} \\
&= \frac{(akt+1)(a^2k)-(a^2kt)(ak)}{(akt+1)^2} \\
&= \frac{a^3k^2t+a^2k-a^3k^2t}{(akt+1)^2} \\
&= \frac{a^2k}{(akt+1)^2}
\end{aligned}
$$

Now we need to show that this is equivalent to $k(a-x)^2$.

$$
\begin{aligned}
k(a-x)^2 &= k\left(a-\frac{a^2kt}{akt+1}\right)^2 \\
&= k\left(a\frac{akt+1}{akt+1}-\frac{a^2kt}{akt+1}\right)^2 \\
&= k\left(\frac{a^2kt+a-a^2kt}{akt+1}\right)^2 \\
&= k\left(\frac{a}{akt+1}\right)^2 \\
&= \frac{a^2k}{(akt+1)^2}
\end{aligned}
$$

So we have shown $dx/dt = k(a-x)^2$.

As $t \to \infty$, we have for the concentration

$$
\begin{aligned}
\lim_{t\to\infty}([\mathrm{C}]) &= \lim_{t\to\infty}\frac{a^2kt}{akt+1} \\
&= \lim_{t\to\infty}\frac{a^2kt\cdot\frac{1}{t}}{(akt+1)\cdot\frac{1}{t}} \\
&= \lim_{t\to\infty}\frac{a^2k}{(ak+\frac{1}{t})} \\
&= \frac{a^2k}{(ak+0)} = \frac{a^2k}{ak} = a
\end{aligned}
$$

The concentration approaches a moles/L.

For the instantaneous rate of reaction we find

$$\lim_{t\to\infty}\left(\frac{d[\mathrm{C}]}{dt}\right) = \lim_{t\to\infty}\frac{a^2k}{(akt+1)^2} = 0$$

The instantaneous rate of reaction approaches 0 moles/L/s.

Practically, this means that the reaction will essentially stop after a long period of time.