Example (4.10.10) Find the most general antiderivative for the function

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6}$$

We cannot find an antiderivative of the function in its present form, so we should use algebra to rewrite the function in a form for which we can get the antiderivative.

$$\begin{aligned} g(x) &= \frac{5 - 4x^3 + 2x^6}{x^6} \\ &= 5x^{-6} - 4x^{-3} + 2 \\ G(x) &= 5 \cdot \frac{1}{-5}x^{-5} - 4 \cdot \frac{1}{-2}x^{-2} + \frac{1}{1}2x + C \\ &= -x^{-5} + 2x^{-2} + 2x + C \\ G'(x) &= -(-5)x^{-6} + 2(-2)x^{-3} + 2(1) \\ &= 5x^{-6} - 4x^{-3} + 2 \\ &= g(x) \end{aligned}$$

Example (4.10.28) Find f when

$$f'(x) = 2x - 3/x^4, x > 0, f(1) = 3.$$

We can get f with a single antidifferentiation. Then we will use the condition $f(1) = 3$ to determine the constant that is introduced.

$$\begin{aligned} f'(x) &= 2x - 3x^{-4} \\ f(x) &= 2 \cdot \frac{1}{2}x^2 - 3 \cdot \frac{1}{-3}x^{-3} + C \\ &= x^2 + x^{-3} + C \\ f(1) &= (1)^2 + (1)^{-3} + C \\ &= 2 + C \\ f(1) &= 3 \\ 3 &= 2 + C \\ 1 &= C \\ f(x) &= x^2 + x^{-3} + 1 \end{aligned}$$

Example (4.10.34) Find f when

$$f''(x) = 4 - 6x - 40x^3, f(0) = 2, f'(0) = 1.$$

We can get f with two antidifferentiations. Then we will use the two conditions to determine the constants that are introduced.

$$f''(x) = 4 - 6x - 40x^3$$

$$\begin{aligned}
f'(x) &= 4x - 6\cdot\frac{1}{2}x^2 - 40\cdot\frac{1}{4}x^4 + C_1 \\
&= 4x - 3x^2 - 10x^4 + C_1 \\
f(x) &= 4\cdot\frac{1}{2}x^2 - 3\frac{1}{3}x^3 - 10\frac{1}{5}x^5 + C_1x + C_2 \\
&= 2x^2 - x^3 - 2x^5 + C_1x + C_2
\end{aligned}$$

$$\begin{aligned}
f'(0) &= 4(0) - 3(0)^2 - 10(0)^4 + C_1 \\
&= C_1 \\
f'(0) &= 1 \\
C_1 &= 1
\end{aligned}$$

$$\begin{aligned}
f(0) &= 2(0)^2 - (0)^3 - 2(0)^5 + (0) + C_2 \\
&= C_2 \\
f(0) &= 2 \\
C_2 &= 2
\end{aligned}$$

$$f(x) = 2x^2 - x^3 - 2x^5 + x + 2$$

Example (4.10.60) A particle is moving with the given data. Find the position of the particle.

$v(t) = 1.5\sqrt{t}, s(4) = 10.$

$$\begin{aligned}
v(t) &= 1.5\sqrt{t} = \frac{3}{2}t^{1/2} \\
s(t) &= \frac{3}{2}\cdot\frac{1}{3/2}t^{3/2} + C \\
&= t^{3/2} + C \\
s(4) &= 4^{3/2} + C \\
&= 8 + C \\
s(4) &= 10 \\
8 + C &= 10 \\
C &= 2 \\
s(t) &= t^{3/2} + 2
\end{aligned}$$

Example (4.10.60) A particle is moving with the given data. Find the position of the particle.

$a(t) = \cos t + \sin t, s(0) = 0, v(0) = 5.$

$$a(t) = \cos t + \sin t$$

$$\begin{aligned} v(t) &= \sin t - \cos t + C_1 \\ s(t) &= -\cos t - \sin t + C_1 t + C_2 \end{aligned}$$

$$\begin{aligned} v(0) &= \sin 0 - \cos 0 + C_1 \\ &= 0 - 1 + C_1 \\ &= -1 + C_1 \\ v(0) &= 5 \\ -1 + C_1 &= 5 \\ C_1 &= 6 \end{aligned}$$

$$\begin{aligned} s(0) &= -\cos 0 - \sin 0 + 6(0) + C_2 \\ &= -1 - 0 + C_2 \\ &= -1 + C_2 \\ s(0) &= 0 \\ -1 + C_2 &= 0 \\ C_2 &= 1 \end{aligned}$$

$$s(t) = -\cos t - \sin t + 6t + 1$$

Example (4.10.74) A car is traveling at 50 mph when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance covered before the car comes to a stop?

The first thing we have to do with this problem is make sure the units are consistent. I choose to work with feet and seconds.
1 h = 60 min = 3600 s .
1 mile = 5280 feet.
50 mph = 50 $\frac{\text{miles}}{\text{hour}}$ = 50 $\frac{5280\text{feet}}{3600\text{s}}$ = 220/3 ft/s.

Now we can start to solve the problem. We know the velocity is 220/3 ft/s when the brakes are applied, and we know the acceleration during the braking period is −22 ft/s^2 (negative because the car is slowing down). We have

$$\begin{aligned} a(t) &= -22 \\ v(t) &= -22t + C_1 \\ s(t) &= -22 \cdot \frac{1}{2} t^2 = -11t^2 + C_1 t + C_2 \end{aligned}$$

Using the initial velocity as 220/3 ft/s, we can determine C_1 (we assume that $t = 0$ is when the brakes are applied):

$$\begin{aligned} v(t) &= -22t + C_1 \\ v(0) &= -22(0) + C_1 = 220/3 \\ v(t) &= -22t + 220/3 \end{aligned}$$

and the position is given by:

$$s(t) = -11t^2 + \frac{220}{3} t + C_2.$$

The distance it takes the car to stop is given by

$$s(t_s) - s(0) = -11t_s^2 + \frac{220}{3}t_s + C_2 + 11(0)^2 - \frac{220}{3}(0) - C_2 = -11t_s^2 + \frac{220}{3}t_s$$

where t_s is the time it takes for the car to stop. Notice that we do not need to know the value of C_2 to solve the problem!

The time it takes the car to stop is determined by when the velocity is zero,

$$v(t_s) = 0 = -22t_s + 220/3,$$

which leads to $t_s = 10/3$.

The distance it takes the car to stop is

$$s(10/3) - s(0) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3}\cdot\frac{10}{3} = \frac{1100}{9} = 122.22 \text{ ft.}$$

This number is called the *braking distance* on the following UK website http://www.hintsandthings.co.uk/garage/stopmph.htm, and they estimate a braking distance of 125 feet when traveling at 50 mph.