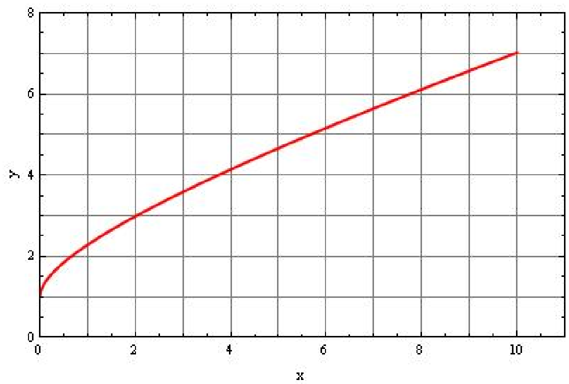
Questions

Example Your numbers probably won't agree exactly with mine, but your analysis should be similar.

By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under f from $x = 0$ to $x = 10$. In each case sketch the rectangles you used. Find new estimates using 10 rectangles in each case.



Example Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

$$f(x) = \frac{\ln x}{x}, 3 \leq x \leq 10.$$

Definition 2 is: The area of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n.$$

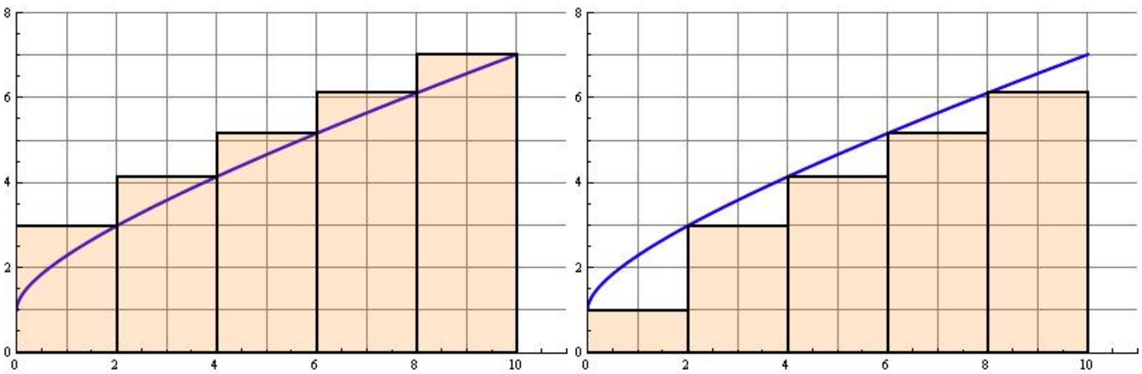
Example Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}.$$

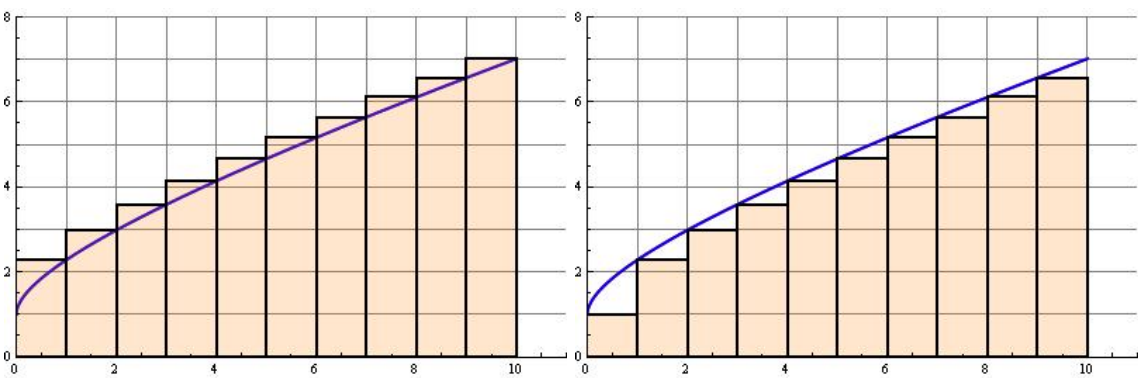
Solutions

Example Your numbers probably won't agree exactly with mine, but your analysis should be similar.

By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under f from $x = 0$ to $x = 10$. In each case sketch the rectangles you used. Find new estimates using 10 rectangles in each case.



The overestimate is: $(2 * 3) + (2 * 4.2) + (2 * 5.2) + (2 * 6.2) + (2 * 7) = 51.2$.
The underestimate is: $(2 * 1) + (2 * 3) + (2 * 4.2) + (2 * 5.2) + (2 * 6.2) = 39.2$.



The overestimate is: $(1*2.2)+(1*3)+(1*3.6)+(1*4.2)+(1*4.6)+(1*5.2)+(1*5.6)+(1*6.2)+(1*6.5)+(1*7) = 48.1$.
The underestimate is: $(1*1)+(1*2.3)+(1*3)+(1*3.6)+(1*4.2)+(1*4.6)+(1*5.2)+(1*5.6)+(1*6.2)+(1*6.5) = 42.2$.

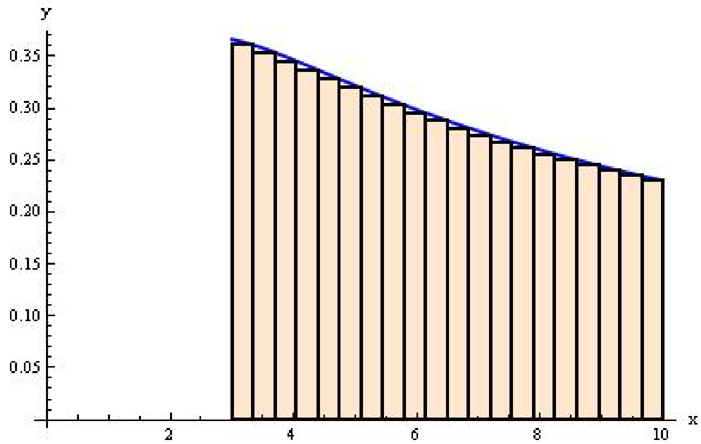
Example Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

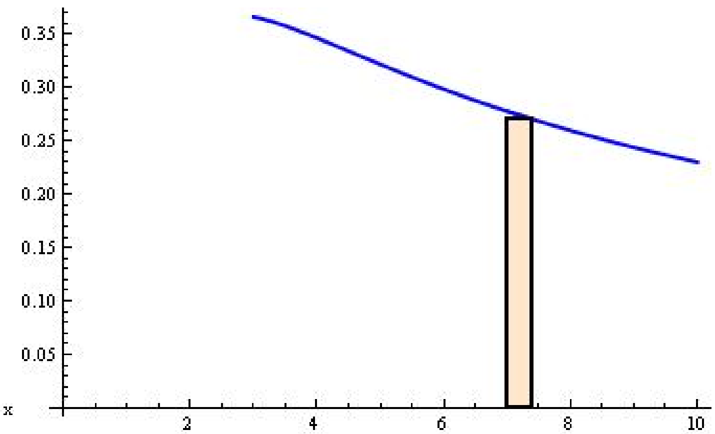
$$f(x) = \frac{\ln x}{x}, 3 \leq x \leq 10.$$

Definition 2 is: The area of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n.$$

The best way to do this is to draw a picture, with a single rectangle that represents all the rectangles. I usually don't draw the sketch on the left, but this time I will. The sketch on the right is used to represent all the rectangles we see in the graph on the left without drawing them.





The height of the rectangle is $f(x_i)$ and the width is Δx. The area of the rectangle is therefore $f(x_i)\Delta x$.
If we have n rectangles, then $\Delta x = (b-a)/n = (10-3)/n = 7/n$.
Since we are evaluating at the right endpoint, we know that $x_i = a + i\Delta x = 3 + 7i/n$.
The area of the rectangle is $f(x_i)\Delta x = \dfrac{\ln(3+7i/n)}{3+7i/n} \cdot \dfrac{7}{n}$.
If we add up the area of all the rectangles we get $\displaystyle\sum_{i=1,n} \frac{\ln(3+7i/n)}{3+7i/n} \cdot \frac{7}{n}$.
If we take the limit as the number of partitions (or rectangles) becomes infinite, we get the area under the curve,

$$Area = \lim_{n\to\infty} \sum_{i=1,n} \frac{\ln(3+7i/n)}{3+7i/n} \cdot \frac{7}{n} = \lim_{n\to\infty} \sum_{i=1,n} \frac{7\ln(3+7i/n)}{3n+7i}.$$

Example Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}.$$

To solve this problem we need to compare to something. Here are some things we might compare to:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{(b-a)}{n}\right)\frac{(b-a)}{n}$$

The last form, although maybe initially the most intimidating, is the most useful.

Here is a colour coded comparison:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{(b-a)}{n}\right)\frac{(b-a)}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$$

The green tells us $b - a = 2$.
The red tells us $b - a = 2$ and $a = 5$. This means $b = 7$.
The blue tells us that $f(x) = x^{10}$.

Putting it all together, we have that the limit is equal to the area under the curve $f(x) = x^{10}$ from $x = 5$ to $x = 7$.