The composition of a function is given by
$y = (f \circ g)(x) = f(g(x))$,
which is decomposed into
$y = f(u), \quad u = g(x)$.
The chain rule is then expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Using this notation allows one to easily create longer chains:
If $y = f(g(h(x)))$ decompose as:
$y = f(u), \quad u = g(w), \quad w = h(x)$.
and the chain rule now has two links in the chain:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$$

Example Given $y = \sqrt{x^2 + 1}$, find y'.

First, we must decompose the function:
Let $y = \sqrt{u}, \quad u = x^2 + 1$,

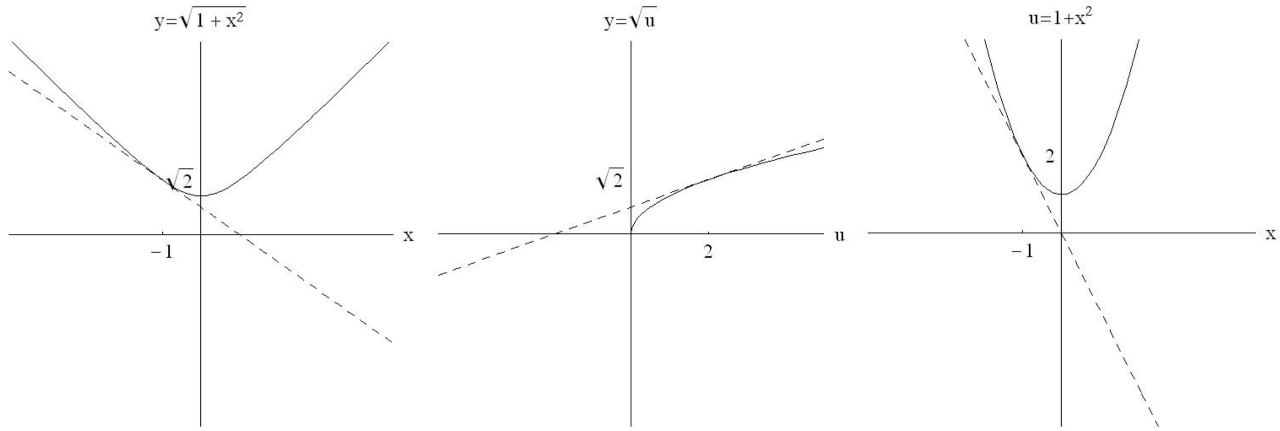
$$\begin{aligned} y' = \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\sqrt{u}]\frac{d}{dx}[x^2 + 1] \\ &= (\frac{1}{2\sqrt{u}})(2x) \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

Exploration What is happening for $y = \sqrt{x^2 + 1}$ when we use the chain rule to calculate the derivative y' at $x = -1$? Let's see what we can learn graphically.

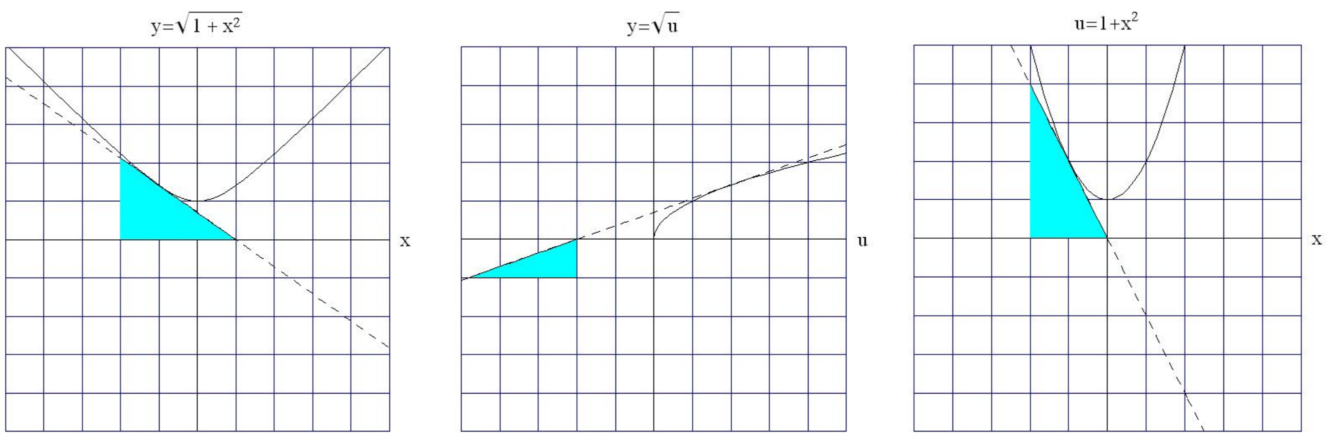
Decompose: $y = \sqrt{u}$, $u = 1 + x^2$. We have three graphs in this problem–notice the importance of labelling the axes–one for $y = \sqrt{1 + x^2}$, one for $y = \sqrt{u}$, and one for $u = 1 + x^2$.

When $x = -1$, $u = 1 + (-1)^2 = 2$. So we can sketch the tangent to the curves at $x = -1$ and $u = 2$, depending on what the independent variable is for the graph.

The graphs on the middle and right are the decomposition of the graph on the left.



If we estimate the slope of the tangent lines to these curves, we can see how the chain rule works. The grid below is in units of 1, and the triangles help us estimate the slope of the tangent lines.



$$\left.\frac{dy}{dx}\right|_{x=-1} \sim -\frac{2}{3} = \left.\frac{dy}{du}\right|_{u=2} \cdot \left.\frac{du}{dx}\right|_{x=-1} \qquad \left.\frac{dy}{du}\right|_{u=2} \sim \frac{1}{3} \qquad \left.\frac{du}{dx}\right|_{x=-1} \sim -\frac{4}{2} = -2$$

We have graphically verified for $y = \sqrt{1+x^2}$ the chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. This is very cool indeed.