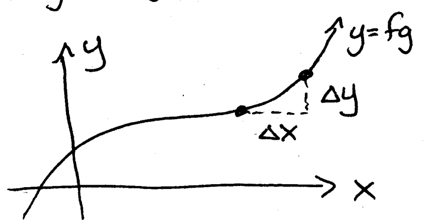
Proof of the product rule of derivatives: $\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$

This can be proven using difference quotients.

Let $y = fg = f(x)g(x)$ (we will suppress the functional notation "of x" for now).



$$y + \Delta y = (f+\Delta f)(g+\Delta g)$$
$$= fg + g\Delta f + f\Delta g + \Delta f\Delta g \quad \text{use } fg = y$$
$$= y + g\Delta f + f\Delta g + \Delta f\Delta g$$

so $\quad \Delta y = g\Delta f + f\Delta g + \Delta f\Delta g$.

Now, divide by Δx, since we want to use $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$.

$$\frac{\Delta y}{\Delta x} = g\frac{\Delta f}{\Delta x} + f\frac{\Delta g}{\Delta x} + \Delta f\frac{\Delta g}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = g\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} + f\lim_{\Delta x\to 0}\frac{\Delta g}{\Delta x} + \lim_{\Delta x\to 0}\Delta f\frac{\Delta g}{\Delta x}$$

as $\Delta x \to 0$, we have $\Delta y \to 0$ and $\Delta f \to 0$ (see diagram and definition for y above).

$$\frac{dy}{dx} = g\frac{df}{dx} + f\frac{dg}{dx} + 0$$

so $\quad \frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$

or $\quad \frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$. Done!

Proof of power rule $\frac{d}{dx}[x^n] = nx^{n-1}$.

This proof uses induction.

a) check power rule is true for $n=1$:

$$\frac{d}{dx}[x^1] = \lim_{h\to 0} \frac{(x+h)-(x)}{h}$$
$$= 1$$
$$= 1 \cdot x^{1-1}$$

so $\frac{d}{dx}[x^n] = nx^{n-1}$ is true for $n=1$.

b) We then assume $\frac{d}{dx}[x^k] = kx^{k-1}$ is true.

c) The induction step is to prove $\frac{d}{dx}[x^{k+1}] = (k+1)x^k$ is true, given a) and b).

$$\frac{d}{dx}[x^{k+1}] = \frac{d}{dx}[x^k \cdot x]$$ use product rule

$$= \frac{d}{dx}[x^k] \cdot x + x^k \frac{d}{dx}[x]$$ use results from a) and b).

$$= kx^{k-1} \cdot x + x^k \cdot 1$$
$$= kx^k + x^k$$
$$= (k+1)x^k$$

so the induction step is true.

therefore, by induction, $\frac{d}{dx}[x^n] = nx^{n-1}$ for $n = 1, 2, 3, 4, \ldots$

Proof of the Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

This can be proven using the definition of derivative.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[s(x)\right] \quad \text{where } s(x) = \frac{f(x)}{g(x)}$$

$$= \lim_{h\to 0} \frac{s(x+h) - s(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)h}$$

add $0 = f(x)g(x) - f(x)g(x)$ to numerator.

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) - g(x+h)f(x) + f(x)g(x)}{g(x+h)g(x)h}$$

(why? Because we want to factor and be left with limits that can be interpreted as derivatives)

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \frac{g(x)}{g(x+h)g(x)} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot \frac{f(x)}{g(x+h)g(x)}$$

g is differentiable, so it is continuous, so $g(x+h) \to g(x)$ as $h \to 0$.

$$= f'(x) \cdot \frac{g(x)}{(g(x))^2} - g'(x)\frac{f(x)}{(g(x))^2}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Done!